Derived categories

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Contents

1 Introduction

The goal of this course is to get to the definition of derived categories, and be able to discuss derived functors in these terms. We will start with additive and abelian categories. Given an abelian category $\mathcal{A}$, we'll consider the category $C(\mathcal{A})$ of cochain complexes in $\mathcal{A}$, which is also abelian.

Then we'll study $K(\mathcal{A})$, the homotopy category of cochain complexes - here the objects are again cochain complexes, but the morphisms are considered only up to chain homotopy. Unfortunately, $K(\mathcal{A})$ is not abelian, but a suitable approximation for this is that $K(\mathcal{A})$ is a triangulated category, where "distinguished triangles" serve as an approximation/replacement for exact sequences.

Next we consider quasi-isomorphisms of chain complexes, which are chain maps such that all the induced maps on homology are isomorphisms. To get from $K(\mathcal{A})$ to the derived category, we formally invert all quasi-isomorphisms. This is akin to the process of localizing a ring, or forming the Grothendieck group of a monoid.

Once we've defined the derived category, we'll try to rework the usual definition of right derived functors in the language of derived categories. Recall that usually describing derived functors involves things like injective resolutions, and there is some hassle of showing that the chosen resolution does not impact the final calculations. In the derived category language, we can avoid some of this hassle, by obtaining a more abstract and "coordinate-free" version of derived functors. This framework has advantages when proving more abstract results regarding derived functors. In particular, we can discuss compositions of derived functors without dealing with the Grothendieck spectral sequence.

2 Abelian categories

Before getting to the definition of an abelian category, we define additive categories. Every abelian category is additive, so this builds to toward abelian categories.

2.1 Additive categories

Definition 2.1. A category $\mathcal{C}$ is **additive** if

1. For any objects A, B, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, and composition of morphisms is bilinear. This means than if we have

   $$A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C \xrightarrow{h} D$$

   then

   $$h \circ (g_1 + g_2) \circ f = h \circ g_1 \circ f + h \circ g_2 \circ f$$

2. $\mathcal{C}$ has a zero object (an object which is both initial and terminal). We denote this object by 0.

3. $\mathcal{C}$ has all finite products and coproducts, and these coincide. These are denoted $A \oplus B$, and sometimes referred to as biproducts.

Remark 2.2. In the definition above, it is sufficient to only assume that all finite coproducts exist. From this and the other properties, it follows that finite products exist, and that they coincide with coproducts.

Definition 2.3. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is **additive** if for any two objects A, B, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB)$$

is a morphism of abelian groups.

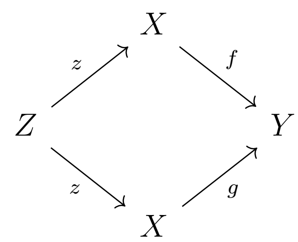
Remark 2.4. It follows from above that an additive functor preserves finite biproducts.

Example 2.5. Let R be a ring (not necessarily commutative), and let $\mathcal{C}$ be the category of left R-modules. Then $\mathcal{C}$ is additive. Similarly, the category of finitely generated left R-modules is additive.

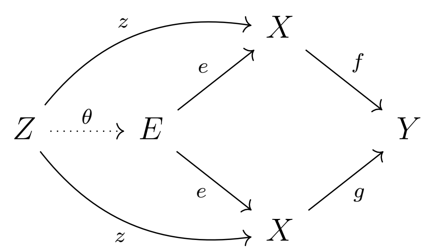
Definition 2.6. Let $\mathcal{C}$ be a category, and let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. The **equalizer** of f and g is an object E and a morphism $e : E \to X$ making the following diagram commute,

$$\begin{array}{ccccc} & & X & & \\ & \overset{e}{\nearrow} & & \overset{f}{\searrow} & \\ E & & & & Y \\ & \underset{e}{\searrow} & & \underset{g}{\nearrow} & \\ & & X & & \end{array}$$

and such that E, e are universal in this diagram. Concretely, that means that for any object Z with a morphism $z : Z \to X$ such that the analogous diagram commutes,



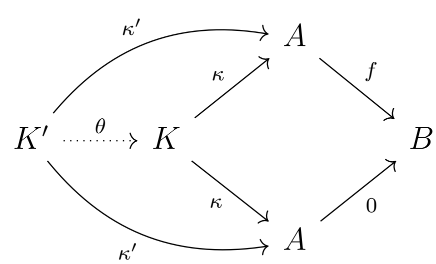
then there exists a unique morphisms $\theta : Z \to E$ making the following diagram commute.



Definition 2.7. The coequalizer of two morphisms is defined analogously, with the arrows reversed.

Remark 2.8. In an abelian category, the equalizer of two morphisms f, g is just the kernel of $f - g$.

Definition 2.9. Let $f : A \to B$ be a morphism in an additive category $\mathcal{C}$. The **kernel** of f is the equalizer of f and the zero morphism $A \xrightarrow{0} B$. That is, $\ker f$ is a pair (K, κ), where $\kappa : K \to A$ is a morphism such that $f\kappa = 0$ and κ is universal with this property. This means that if $\kappa' : K' \to A$ also satsfies $f\kappa' = 0$, then there is a unique morphism $\theta : K' \to K$ such that $\kappa' = \kappa\theta$.



Definition 2.10. Let $f : A \to B$ be a morphism in an additive category. The **cokernel** of f is the coequalizer of f and the zero morphism.

Remark 2.11. While technically a kernel or cokernel is a pair of an object with a morphism, frequently we are loose with language and refer to either the associated morphism or the associated object as the kernel. In concrete categories like R-mod, it's more common to think of the associated object as the kernel, since the associated morphism is just an inclusion map. In more abstract/general settings, often the morphism is the focus.

Definition 2.12. Let $\mathcal{C}$ be a category. The **opposite category** C^{op} has the same objects as $\mathcal{C}$, but the arrows are all reversed.

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) := \text{Hom}_{\mathcal{C}}(B, A)$$

Example 2.13. Given a morphism f in $\mathcal{C}$, the kernel of f in $\mathcal{C}$ is the same as $\operatorname{coker} f$ in $\mathcal{C}^{\text{op}}$.

Definition 2.14. A morphism $f : X \to Y$ is a **monomorphism** if for any $g_1, g_2 : Z \to X$ we have

$$fg_1 = fg_2 \implies g_1 = g_2$$

$$Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

Definition 2.15. A morphism $f : X \to Y$ is an **epimorphism** if for any $h_1, h_2 : Y \to Z$ we have

$$h_1 f = h_2 f \implies h_1 = h_2$$

$$X \xrightarrow{f} Y \underset{h_2}{\overset{h_1}{\rightrightarrows}} Z$$

Remark 2.16. Let $f : X \to Y$. The kernel of f, if it exists, is a monomorphism. The cokernel, if it exists, is an epimorphism.

Kernels and cokernels may fail to exist in additive categories.

Example 2.17 (Failure of kernel to exist)**.** Let R be a non-Noetherian commutative ring, such as $k[x_1, x_2, \ldots]$, a polynomial ring in countably many variables over a field. Let $\mathcal{C}$ be the category of finitely generated R-modules. As R is non-Noetherian, it has an ideal I which is not finitely generated (as an ideal, so also not finitely generated as an R-module). Consider the quotient morphism

$$R \xrightarrow{f} R/I$$

In the category of R-modules, the kernel of f is I, but this cannot be the kernel in the category of finitely generated R-modules, so in fact this morphism has no kernel in $\mathcal{C}$.

Example 2.18 (Failure of cokernel to exist)**.** Let $\mathcal{C}$ be the category of free abelian groups. This is an additive category. Consider the morphism

$$\mathbb{Z} \to \mathbb{Z} \qquad n \mapsto 2n$$

The cokernel "should be" $\mathbb{Z}/2\mathbb{Z}$ (this is the cokernel in the category of abelian groups), but this is not free. So this morphism fails to have a cokernel in $\mathcal{C}$.

These examples motivate the definition of abelian categories, where such "bad behavior" is prohibited.

2.2 Abelian categories

Definition 2.19. A category $\mathcal{A}$ is **abelian** if it is additive, and satisfies

1. Every morphism in $\mathcal{A}$ has a kernel and cokernel.
2. Every monomorphism is the kernel of its cokernel.
3. Every epimorphism is the cokernel of its kernel.

Remark 2.20. This definition is self-dual, so $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{\text{op}}$ is abelian.

Remark 2.21. How should we think about properties (2) and (3) above? As a slogan, we think of them as saying

The first isomorphism theorem holds in $\mathcal{A}$.

Don't take this too literally, but this is roughly what properties (2) and (3) are trying to capture. Below, we reformulate these properties in terms of "strict" morphisms.

Definition 2.22. Let $f : X \to Y$ be a morphism in an abelian category $\mathcal{A}$. Let $\kappa : K \to X$ and $\rho : Y \to Q$ be the kernel and cokernel of f, respectively.

$$K \xrightarrow{\kappa} X \xrightarrow{f} Y \xrightarrow{\rho} Q$$

We can factor f through the cokernel of κ.

$$X \xrightarrow{\alpha} \operatorname{coker} \kappa \xrightarrow{\beta} Y, \qquad f = \beta\alpha$$

Then $0 = \rho f = \rho\beta\alpha$. Since α is the canonical map associated with the cokernel of κ, it is an epimorphism. So $\rho\beta\alpha = 0 \implies \rho\beta = 0$. Then by the universal property of the kernel of ρ, β factors through $\ker \rho$.

$$\begin{array}{ccccccc} K & \xrightarrow{\kappa} & X & \xrightarrow{f} & Y & \xrightarrow{\rho} & Q \\ & & \Big\downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle \beta} & \Big\uparrow & & \\ & & \operatorname{coker} \kappa & \xrightarrow{\iota} & \ker \rho & & \end{array}$$

In the diagram above, the arrow $\ker \rho \to Y$ is the canonical map associated with the kernel. The morphism $\iota : \operatorname{coker} \kappa \to \ker \rho$ is called the **canonical map associated to** f. To write things just in terms of f,

$$\iota : \operatorname{coker} \ker f \to \ker \operatorname{coker} f$$

Definition 2.23. The morphism f is called **strict** if the associated map ι is an isomorphism.

Proposition 2.24. *Every morphism in an abelian category is strict.*

We will get to the proof later. In fact, one can show more than the proposition - it is possible to show that properties (2) and (3) of an abelian category are equivalent to every morphism being strict, but we will not concern ourselves with the reverse implication. The proof will proceed in three main steps.

1. Show the map $\beta : \operatorname{coker}\ker f \to Y$ is a monomorphism.
2. Show $\operatorname{coker}\beta = \operatorname{coker} f$.
3. Use the fact that every monomorphism is the kernel of its cokernel.

Before the proof, we develop some terminology and a lemma.

Definition 2.25. Let $f : A \to C, g : B \to C$ be morphisms in a category. The **pullback** of f and g is the limit of the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

The object associated with the pullback is denoted $A \times_C D$. This notation is sometimes unfortunate, since it does not emphasize how the pullback depends heavily on the morphisms f and g.

Lemma 2.26 (Pullbacks in abelian categories)**.** *Let $\mathcal{A}$ be an abelian category.*

1. *Pullbacks exist in $\mathcal{A}$.*
2. *Epimorphisms are stable under pullback in $\mathcal{A}$. That is, if*

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

is a pullback square and g is an epimorphism, then h is an epimorphism.

Remark 2.27. In an algebraic geometry context, part (2) of the previous lemma might be phrased as saying that "epimorphisms are stable under base extension."

Proof. (1) Let $A \oplus B$ be the biproduct of A, B, with canonical projection maps $p_1 : A \oplus B \to A, p_2 : A \oplus B \to B$. Consider the map

$$fp_1 - gp_2 : A \oplus B \to C$$

Let $\kappa : D \to A \oplus B$ be the kernel of the above morphism. Set $h = p_1\kappa : D \to A$ and $k = p_2\kappa : D \to B$. Then

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is the pullback. This technically requires verifying a universal property, which we omit.

(2) Now suppose g is an epimorphism. First, we will show that $fp_1 - gp_2$ is epi. Let $\phi_1, \phi_2 : C \to Z$ be maps such that

$$\phi_1(fp_1 - gp_2) = \phi_2(fp_1 - gp_2)$$

We wish to show $\phi_1 = \phi_2$. Let $i_2 : B \to A \oplus B$ be the canonical map associated with the coproduct, so we know $p_1 i_2 = 0$ and $p_2 i_2 = \mathrm{Id}_B$. Then

$$\phi_1(fp_1 - gp_2)i_2 = \phi_1 f_1 p_1 i_2 - \phi_1 g p_2 i_2 = -\phi_1 g$$

Similarly,

$$\phi_2(fp_1 - gp_2)i_2 = -\phi_2 g$$

Thus $\phi_1 g = \phi_2 g$. Since g is epi, $\phi_1 = \phi_2$, so $fp_1 - gp_2$ is epi as claimed. Now we can show h is epi. Suppose $\psi_1, \psi_2 : A \to Y$ are morphisms such that $\psi_1 h = \psi_2 h$. Then $(\psi_1 - \psi_2)h = 0$. By part (1), $h = p_1 \kappa$ where $\kappa : D \to A \oplus B$ is the kernel of $fp_1 - gp_2$. So

$$(\psi_1 - \psi_2)p_1 \kappa = 0$$

Thus $(\psi_1 - \psi_2)p_1$ factors through $\operatorname{coker} \kappa$. But we showsed that $fp_1 - gp_2$ is epi, and κ is its kernel. So $fp_1 gp_2$ is the cokernel of κ, by the defining properties of an abelian category. So $(\psi_1 - \psi_2)p_1$ factors through C, meaning there is a morphism $t : C \to Y$ making the following diagram commute.

$$\begin{array}{ccc} A \oplus B & \xrightarrow{fp_1 - gp_2} & C \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle t} \\ A & \xrightarrow{\psi_1 - \psi_2} & Y \end{array}$$

Now consider the composition with either path on the above square with $i_2 : B \to A \oplus B$.

$$0 = (\psi_1 - \psi_2)p_1 i_2 = t(fp_1 - gp_2)i_2 = tfp_1 i_2 - tgp_2 i_2 = -tg$$

So $-tg = 0$. Since g is epi, this implies $t = 0$. Then by the commutative square, $(\psi_1 - \psi_2)p_1 = 0$. Since p_1 is epi, this means $\psi_1 - \psi_2 = 0$, so $\psi_1 = \psi_2$. Hence h is epi. □

Remark 2.28. The dual argument shows that pushouts exist in abelian categories.

Remark 2.29. Part (1) of the lemma and the previous remark are significantly generalized by the following fact: in an abelian category, all finite limits and colimits exist.

Next we state a lemma without proof.

Lemma 2.30. *Let $f : X \to Y$ be a morphism in an abelian category $\mathcal{A}$ with factorization $X \xrightarrow{g} X' \xrightarrow{h} Y$ where g is an epimorphism. Then* $\operatorname{coker} f \cong \operatorname{coker} h$.

Now we have the tools to prove the prosition from before.

Proposition 2.31. *Every morphism in an abelian category is strict.*

As mentioned, the proof will follow this outline:

1. Show the map $\beta : \operatorname{coker}\ker f \to Y$ is a monomorphism.
2. Show $\operatorname{coker}\beta = \operatorname{coker} f$.
3. Use the fact that every monomorphism is the kernel of its cokernel.

Proof. Let $\kappa, \rho, \alpha, \beta, \iota$ be as in definition 2.22.

$$\begin{array}{ccccccc} K & \xrightarrow{\kappa} & X & \xrightarrow{f} & Y & \xrightarrow{\rho} & Q \\ & & \downarrow{\scriptstyle\alpha} & \nearrow{\scriptstyle\beta} & \uparrow & & \\ & & \operatorname{coker}\kappa & \xrightarrow{\iota} & \ker\rho & & \end{array}$$

(1) We show β is a monomorphism. Suppose $\phi_1, \phi_2 : Z \to \operatorname{coker}\kappa$ satisfy $\beta\phi_1 = \beta\phi_2$. Let $\sigma = \phi_1 - \phi_2$, so $\beta\sigma = 0$. Consider the pullback of α and σ.

$$\begin{array}{ccc} W & \xrightarrow{\tau} & Z \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow{\alpha} & \operatorname{coker}\kappa \end{array}$$

Then since $f = \beta\alpha$,

$$f\pi = \beta\alpha\pi = \beta\sigma\tau = 0$$

Thus π factors through $\ker f = K$. That is, there exists a unique morphism $\omega : W \to K$ making the following diagram commute.

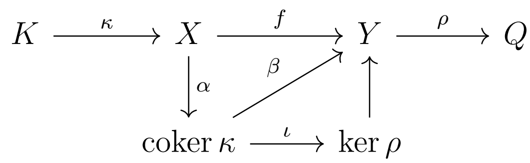
$$\begin{array}{ccccc} & & W & \xrightarrow{\tau} & Z \\ & \swarrow{\scriptstyle\omega} & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\sigma} \\ K & \xrightarrow{\kappa} & X & \xrightarrow{\alpha} & \operatorname{coker}\kappa \end{array}$$

By definition of the kernel, $\alpha\kappa = 0$, so $\sigma\tau = \alpha\kappa\omega = 0$. Since α is epi, τ is epi by Lemma 2.26. So $\sigma\tau = 0$ implies $\sigma = 0$. Hence $\phi_1 = \phi_2$, and β is a monomorphism.

(2) Now recall that f factors as

$$X \xrightarrow{\alpha} \operatorname{coker}\kappa \xrightarrow{\beta} Y, \qquad f = \beta\alpha : X \to Y$$

Since α is epi, by Lemma 2.30 $\operatorname{coker}\beta = \operatorname{coker} f$. Since β is a monomorphism, it is the kernel of its cokernel. That is, β is the kernel of the cokernel of f. But $\ker\rho \to Y$ is also the kernel of the cokernel of f, so the morphism ι must be an isomorphim by the universal property of the kernel.



$\square$

Proposition 2.32. *Let $\mathcal{A}$ be an abelian category and let $f : X \to Y$ be a morphism in $\mathcal{A}$.*

1. *f is a monomorphism iff $\ker f = 0$.* [1]
2. *f is an epimorphism iff $\operatorname{coker} f = 0$.*
3. *f is an isomorphism iff it is a monomorphism and an epimorphism.*

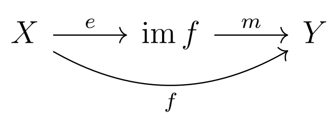
Proof. (1) Suppose f is a monomorphism. We will show that $\ker f$ is the terminal object. Let $\kappa : \ker f \to X$ be the canonical morphism, and let A be an object in $\mathcal{A}$. Suppose $g : A \to \ker f$ be a morphism, and consider the composition $f\kappa g : A \to Y$.

$$A \xrightarrow{g} \ker f \xrightarrow{\kappa} X \xrightarrow{f} Y$$

Since $f\kappa = 0$, $f\kappa g = 0$. Then since f is mono, $\kappa g = 0$, and since κ is mono, $g = 0$. Hence there is only the zero morphism $A \to \ker f$, so $\ker f$ is terminal, hence it is the zero object.

(2) Dual argument to (1).

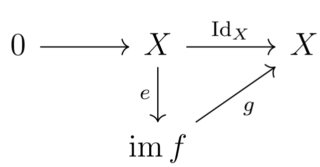
(3) Clearly if f is an isomorphism it can be right- and left-cancelled, so it is mono and epi. Conversely suppose f is mono and epi. Write f in terms of the canonical factorization



Clearly if e and m are isomorphisms, then so is f. We will prove e is an isomorphism, the argument for m is analogous. By construction, e is the cokernel of the kernel of f. By part (1), f is mono. Obviously the composition

$$0 \longrightarrow X \xrightarrow{\mathrm{Id}_X} X$$

is zero. So Id_X factors through the $\operatorname{im} f$. That is, there exists a unique morphism $g : \operatorname{im} f \to X$ such that $ge = \mathrm{Id}_X$.

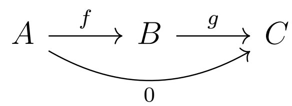


Hence $ege = e$. Since we know e is epi, we can cancel it on the right, so $eg = \mathrm{Id}_{\operatorname{im} f}$, which means e, g are inverses, hence isomorphisms. As mentioned before, a similar/dual argument shows m is an isomorphism. Hence f is an isomorphism. $\square$

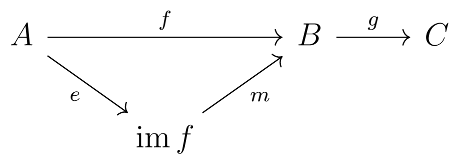
[1] For this and the following parts, $\ker f = 0$ means the object associated with the kernel/cokernel is the zero object.

2.3 Exact sequences

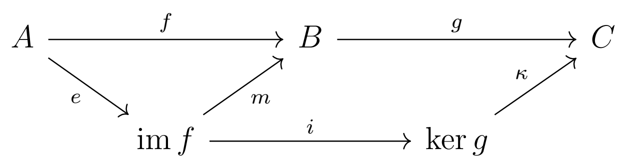
Definition 2.33. Let $\mathcal{A}$ be an abelian category, and suppose that in $\mathcal{A}$ we have morphisms $f : A \to B, g : B \to C$ such that $gf = 0$.



We still have our canonical factorization of f through $\operatorname{im} f$.

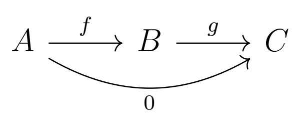


So $gf = gme = 0$. Since e is epi, $gm = 0$. So m factors through $\ker g$, which is to say, there exists a unique map $i : \operatorname{im} f \to \ker g$ and the following diagram commutes.



Given f, g as above such that $gf = 0$, we call i the **canonical map** from $\operatorname{im} f$ to $\ker g$.

Definition 2.34. Let $\mathcal{A}$ be an abelian category. A sequence in $\mathcal{A}$ of the form



with $gf = 0$ is **exact** at B if the canonical map $i : \operatorname{im} f \to \ker g$ is an isomorphism. Similarly, if

$$\cdots \to A_{n-1} \to A_n \to A_{n+1} \to \cdots$$

is a sequence such that every sequential composition of two maps is zero, it is called **exact** if every three-term subsequence is exact. A sequence as above where the index runs over $\mathbb{Z}$ is called a **long exact sequence**. A **short exact sequence** is an exact sequence of the form

$$0 \to A \to B \to C \to 0$$

Remark 2.35. A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism. Similarly, $B \xrightarrow{g} C \to 0$ is exact if and only if g is an epimorphism.

Definition 2.36. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant additive functor between abelian categories.

1. F is **left exact** if for any exact sequence $0 \to A \to B \to C$ in $\mathcal{A}$ the sequence $0 \to FA \to FB \to FC$ is exact in $\mathcal{B}$.

2. F is **right exact** if for any exact sequence $A \to B \to C \to 0$ the sequence $FA \to FB \to FC \to 0$ is exact.

3. F is **exact** if it is right and left exact.

Definition 2.37. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let X, Y be objects in $\mathcal{A}$. Then associated to F is a group homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(FX, FY)$$

The functor F is **faithful** if this map is injective for all objects X, Y. F is **full** if this map is surjective for all objects X, Y. F is **fully faithful** if it is both full and faithful.

Theorem 2.38 (Freyd-Mitchell Embedding Theorem)**.** *Let $\mathcal{A}$ be a small abelian category. Then there exists a unital associative ring R and a fully faithful functor F from $\mathcal{A}$ to the category of left R-modules. That is, $\mathcal{A}$ "embeds" in R-mod.*

Remark 2.39. This theorem is useful even when dealing with abelian categories which are not small. For example, when trying to prove some property of a commutative diagram in a general abelian category, as long as the collection of objects involved in the diagram is not a proper class, one can embed the subcategory of $\mathcal{A}$ consisting of those objects into a category of R-modules using Freyd-Mitchell, and essentially treat the diagram as if it was over the category of R-modules, for the purposes of doing diagram chases, etc.

However, one thing to watch out for is that a fully faithful functor as in the theorem need not preserve injective or projective objects. Essentially, this is because injectives and projectives depend on "global" properties of the category. So it is still important to be able to work in a general abelian category when one wants to work with injective and projective resolutions, for example.

One application of the Freyd-Mitchell embedding theorem is to transfer commonly known facts about the category of R-modules to a general abelian category, especially when the traditional proofs involve diagram chasing. On the other hand, it can be nice to work out more "abstract" proofs of such things in the spirit of a general abelian category. For the next two lemmas, we omit the abstract approach, just relying on Freyd-Mitchell and the traditional result for the category of R-modules.

Lemma 2.40 (5-lemma)**.** *Suppose we have a commutative diagram in an abelian category with exact rows.*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

If f_2, f_4 are isomorphisms, f_1 is epi, and f_5 is mono, then f_3 is an isomorphism.

Lemma 2.41 (Snake lemma)**.** *Suppose we have a commutative diagram in an abelian category with exact rows.*

$$\begin{array}{ccccccccc} & & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & & \end{array}$$

Then there exists a morphism $\partial : \ker h \to \ker f$ *making an exact sequence*

$$\ker f_1 \to \ker f_2 \to \ker f_3 \xrightarrow{\partial} \operatorname{coker} f_1 \to \operatorname{coker} f_2 \to \operatorname{coker} f_3$$

Moreoever, if $A_1 \to A_2$ *is mono, then* $\ker f_1 \to \ker f_2$ *is mono, and if* $B_2 \to B_3$ *is epi, then* $\operatorname{coker} f_2 \to \operatorname{coker} f_3$ *is epi. That is, if the original diagram extends to a larger diagram with exact rows as below,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

then the exact sequence extends to a larger exact sequence

$$0 \to \ker f_1 \to \ker f_2 \to \ker f_3 \xrightarrow{\partial} \operatorname{coker} f_1 \to \operatorname{coker} f_2 \to \operatorname{coker} f_3 \to 0$$

2.4 Chain complexes

Definition 2.42. Let $\mathcal{A}$ be an abelian category. A **cochain complex** in $\mathcal{A}$ is a diagram

$$\cdots \to A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \to \cdots$$

where $i \in \mathbb{Z}$, and $d_A^2 = 0$. That is, $d_a^i d_A^{i-1} = 0$ for all $i \in \mathbb{Z}$. We denote such an object by $A^\bullet = (A^i, d_A^i)$. The morphisms d_A^i are called the **differentials** of $A^\bullet$. A **chain complex** is the same, except the differentials decrease degree instead of increase. We'll mostly focus on cochain complexes, but the difference is immaterial. And actually, because I'm lazy, I'll probably just call both of them chain complexes when I feel like it.

Definition 2.43. Let $A^\bullet, B^\bullet$ be cochain complexes. A **morphism of cochain complexes** or **chain map** is a series of maps $f^i : A^i \to B^i$ making an infinite commutative diagram that looks sort of like a ladder.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \longrightarrow & B^{i-1} d_B^{i-1} & \xrightarrow{d_B^i} & B^i & \xrightarrow{d_B^{i+1}} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

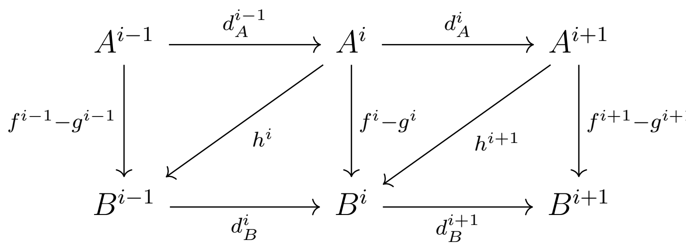
A chain map as above is denoted $f^\bullet$.

Definition 2.44. If $\mathcal{A}$ is a category, cochain complexes with objects from $\mathcal{A}$ along with chain maps form a category denoted $C(\mathcal{A})$. This is called the **category of complexes in** $\mathcal{A}$.

Definition 2.45. Let $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be chain maps. A **chain homotopy** between f and g is a collection of maps $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = d_B^{i-1} h^i + h^{i+1} d_A^i \qquad \forall n \in \mathbb{Z}$$

We include the following diagram (which is NOT commutative) to depict all the maps involved in the equation above, and give some visual ideal of what is going on with a chain homotopy.



The equation says that in the diagram above, going straight down from A^i to B^i via $f^i - g^i$ is the same as the sum of the two triangles involving the h maps.

Definition 2.46. Let $A^\bullet, B^\bullet$ be chain complexes. A **chain homotopy equivalence** is a chain map $f^\bullet : A^\bullet \to B^\bullet$ such that there exists a chain map $g^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet f^\bullet$ is chain homotopic to $\mathrm{Id}_{A^\bullet}$ and $f^\bullet g^\bullet$ is chain homotopic to $\mathrm{Id}_{B^\bullet}$. In such a situation we say $A^\bullet, B^\bullet$ are **chain homotopy equivalent**, and say that $f^\bullet, g^\bullet$ are **homotopy psuedo-inverses**.

Proposition 2.47. *Let $\mathcal{A}$ be a category.*

1. *If $\mathcal{A}$ is additive, then $C(\mathcal{A})$ is additive.*
2. *If $\mathcal{A}$ is abelian, then $C(\mathcal{A})$ is abelian.*

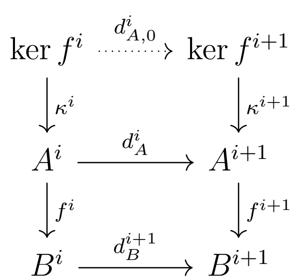
Proof. (1) Just a mostly tedious exercise.

(2) Also mostly a tedious exercise. Instead of a full proof, we give a construction of a kernel in $C(\mathcal{A})$. The basic idea is not surprising - just take the kernel at each step of a chain map, and the resulting sequence of kernels forms a cochain complex which is the kernel in $C(\mathcal{A})$. The details take some working out, though, so we do that here.

Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map. Let $\kappa^i : \ker f^i \to A^i$ be the kernel at each step. We know $f^i \kappa^i = 0$, and then using the fact that f is a chain map,

$$0 = d_B^i f^i \kappa^i = f^{i+1} d_A^i \kappa^i$$

Thus, $d_A^i \kappa^i$ factors through $\ker f^{i+1}$. That is, there exists a unique morphism $d_{A,0}^i : \ker f^i \to \ker f^{i+1}$ making the following diagram commute.



Also, since κ^i is a monomorphism, $d_{A,0}^{i+1} d_{A,0}^i = 0$ for all i. Hence the stepwise kernels form a complex, and it is immediate that $\kappa^\bullet : \ker f^\bullet \to A^\bullet$ is a chain map. After doing this, one should check that the complex constructed above satisfies the universal property of the kernel in $C(\mathcal{A})$, but this is tedious so we omit it. $\square$

Remark 2.48. Suppose $\mathcal{A}$ is an abelian category. There is an exact, fully faithful functor $\mathcal{A} \to C(\mathcal{A})$ which takes an object X of A to the cochain complex with X in degree zero and zeros elsewhere.

$$X \qquad \rightsquigarrow \qquad C(X) = \cdots \to 0 \to X \to 0 \to \cdots$$

This functor takes a morphism $f : X \to Y$ to the obvious chain map $C(X) \to C(Y)$, which is just f in degree zero.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

All unlabelled maps above are the zero map.

2.4.1 Cohomology of chain complexes

In the category of modules over a ring R, given a complex $A^\bullet$, we just define the cohomology to be a certain quotient.

$$H^n(A^\bullet) := \frac{\ker d_A^n}{\operatorname{im} d_A^{n-1}}$$

For a more general abelian category setting, we need some more finesse, since quotients are less clear. But essentially, a quotient is just a cokernel, so the following definition isn't too surprising.

Definition 2.49. Let $A^\bullet$ be a cochain complex in an abelian category $\mathcal{A}$. Since $d_A^n d_A^{n-1} = 0$, we obtain a unique morphism $i_{n-1} : \operatorname{im} d_A^{n-1} \to \ker d_A^n$ from the universal property of the kernel. We define the n**th cohomology object** of $A^\bullet$ to be the (object associated to) the cokernel of i_{n-1}.

$$H^n(A^\bullet) := \operatorname{coker} i_{n-1}$$

Definition 2.50. A complex $A^\bullet$ is **acyclic** if $H^n(A^\bullet) = 0$ for all n. This is equivalent to $A^\bullet$ being exact.

Lemma 2.51 (Functoriality of H^n)**.** *Suppose $f^\bullet : A^\bullet \to B^\bullet$ is a chain map between complexes with objects from an abelian category. Then $f^\bullet$ induces morphisms*

$$H^n(f^\bullet) : H^n(A^\bullet) \to H^n(B^\bullet)$$

If $g^\bullet : B^\bullet \to C^\bullet$ is another chain map, then

$$H^n(g^\bullet \circ f^\bullet) = H^n(g^\bullet) \circ H^n(f^\bullet)$$

Hence $H^n : C(\mathcal{A}) \to \mathcal{A}$ is an additive functor.

Proof. We just give a sketch of the construction of $H^n(f^\bullet)$, and omit proof of the functoriality and additvity. From the fact that f is a chain map, it induces $f_0^n : \ker d_A^n \to \ker d_B^n$ and $f_1^n : \operatorname{im} d_A^{n-1} \to \operatorname{im} d_B^{n-1}$ making the following diagram commute.

$$\begin{array}{ccc} \operatorname{im} d_A^{n-1} & \xrightarrow{i_{A,n-1}} & \ker d_A^n \\ \Big\downarrow f_1^n & & \Big\downarrow f_0^n \\ \operatorname{im} d_B^{n-1} & \xrightarrow{i_{B,n-1}} & \ker d_B^n \end{array}$$

Let $\rho_{B,n} : \ker d_B^n \to \operatorname{coker} i_{B,n-1} = H^n(B^\bullet)$ be the associated canonical map of the cokernel. Using the diagram above,

$$\rho_{B,n} f_0^n i_{A,n-1} = \rho_{B,n} i_{B,n-1} f_1^n$$

Since $\rho_{B,n}$ is the cokernel map of $i_{B,n-1}$, the right side above is zero. Then the left side also vanishes, implying that $f_0^n i_{A,n-1}$ factors through the cokernel, which is $H^n(B^\bullet)$. That is, there is a unique morphism $H^n(f^\bullet) : H^n(A^\bullet) \to H^n(B^\bullet)$ such that the following diagram commutes.

$$\begin{array}{ccccc} \operatorname{im} d_A^{n-1} & \xrightarrow{i_{A,n-1}} & \ker d_A^n & \xrightarrow{\rho_{A,n}} & H^n(A^\bullet) = \operatorname{coker} \rho_{A,n} \\ \Big\downarrow f_1^n & & \Big\downarrow f_0^n & & \Big\downarrow H^n(f^\bullet) \\ \operatorname{im} d_B^{n-1} & \xrightarrow{i_{B,n-1}} & \ker d_B^n & \xrightarrow{\rho_{B,n}} & H^n(B^\bullet) = \operatorname{coker} \rho_{B,n} \end{array}$$

□

Definition 2.52. Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map. It is a **quasi-isomorphism** if all induced morphisms $H^n(f^\bullet)$ are isomorphisms.

Remark 2.53. Let $\mathcal{A}$ be an abelian category. Consider a sequence

$$0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$$

in $C(\mathcal{A})$. Since $C(\mathcal{A})$ is abelian, we can speak of this sequence being exact (or not). It is a short lemma to prove that the sequence above is exact (in $C(\mathcal{A})$) if and only if the associated sequence

$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

is exact (in $\mathcal{A}$) for every n.

Theorem 2.54 (LES in cohomology associated to SES of chain complexes)**.** *Let $\mathcal{A}$ be an abelian category and suppose we have a short exact sequence in $C(\mathcal{A})$.*

$$0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$$

For each n, there exists a morphism $\partial^n : H^n(C^\bullet) \to H^{n+1}(A^\bullet)$ making the following long exact sequence in $\mathcal{A}$.

$$\cdots \to H^n(A^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(B^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(C^\bullet) \xrightarrow{\partial^n} H^{n+1}(A^\bullet) \xrightarrow{H^{n+1}(f^\bullet)} H^{n+1}(B^\bullet) \to \cdots$$

Proof. Essentially we apply the snake lemma twice. We just sketch the argument. First, we have the commutative diagram below which has exact rows, for every i.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^i & \xrightarrow{f^i} & B^i & \xrightarrow{g^i} & C^i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_A^i} & & \downarrow{\scriptstyle d_B^i} & & \downarrow{\scriptstyle d_C^i} & & \\
0 & \longrightarrow & A^{i+1} & \xrightarrow{f^{i+1}} & B^{i+1} & \xrightarrow{f^{i+1}} & C^{i+1} & \longrightarrow & 0
\end{array}$$

Applying the snake lemma to this, we obtain two exact sequences (ignoring the connecting homomorphism).

$$0 \to \ker d_A^i \to \ker d_B^i \to \ker d_C^i \qquad \operatorname{coker} d_A^i \to \operatorname{coker} d_B^i \to \operatorname{coker} d_C^i \to 0$$

Using the fact that $d^2 = 0$ and some universal properties, the differential maps from $A^\bullet, B^\bullet, C^\bullet$ induce maps $\widetilde{d_A^n}, \widetilde{d_B^n}, \widetilde{d_C^n}$ fitting into the following commutative diagram, which has exact rows by the previous application of the snake lemma.

$$\begin{array}{ccccccccc}
 & & \operatorname{coker} d_A^{n-1} & \longrightarrow & \operatorname{coker} d_B^{n-1} & \longrightarrow & \operatorname{coker} d_C^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{d_A^n}} & & \downarrow{\scriptstyle \widetilde{d_B^n}} & & \downarrow{\scriptstyle \widetilde{d_C^n}} & & \\
0 & \longrightarrow & \ker d_A^{n+1} & \longrightarrow & \ker d_B^{n+1} & \longrightarrow & \ker d_C^{n+1} & &
\end{array}$$

Also, one can check that $H^n(A^\bullet) \cong \ker \widetilde{d_A^n}$. So applying the snake lemma to the above, we get the required long exact sequence. □

2.4.2 Translation and truncation

Fix an abelian category $\mathcal{A}$.

Definition 2.55.

1. A complex $A^\bullet$ is **bounded below** if there exists $N \in \mathbb{Z}$ such that $A^i = 0$ for all $i < N$. Complexes which are bounded below form a full subcategory $C^+(\mathcal{A})$ of $C(\mathcal{A})$.

2. A complex $A^\bullet$ is **bounded above** if there exists $N \in \mathbb{Z}$ such that $A^i = 0$ for all $i > N$. Complexes which are bounded above form a full subcategory $C^-(\mathcal{A})$ of $C(\mathcal{A})$.

3. A complex $A^\bullet$ is **bounded** if it is bounded both above and below. Complexes which are bounded form a full subcategory $C^b(\mathcal{A})$ of $C(\mathcal{A})$.

Definition 2.56. The **translation functor** $T : C(\mathcal{A}) \to C(\mathcal{A})$ sends a complex $A^\bullet = (A^i, d_A^i)$ to another complex $T(A^\bullet) = (T(A^\bullet)^i, d^i_{T(A^\bullet)})$ defined by

$$T(A^\bullet)^i = A^{i+1} \qquad d^i_{T(A^\bullet)} = -d_A^{i+1}$$

Essentially, T translates the complex left by one degree. The negative sign in on the translated differentials is just a convention which will be useful later when dealing with cones of morphisms. On a chain map $f^\bullet : A^\bullet \to B^\bullet$, the functor T outputs the chain map

$$T(f^\bullet) : T(A^\bullet) \to T(B^\bullet) \qquad T(f^\bullet)^i = f^{i+1}$$

For situations in which we apply the translation functor repeatedly, we define

$$A^\bullet[n] := T^n(A^\bullet)$$

which gives a convenient way to describe shifting A to the left by an arbitrary number of degrees.

Remark 2.57. Taking cohomology commutes with translation, which is to say, $H^n(T(A^\bullet)) = H^{n+1}(A^\bullet)$.

Definition 2.58. Let $A^\bullet$ be a complex with objects from the abelian category $\mathcal{A}$, and fix $n \in \mathbb{Z}$. The **right truncation of $A^\bullet$ at n** is the complex $\tau_{\leq n}(A^\bullet)$ defined by

$$\tau_{\leq n}(A^\bullet)^i = \begin{cases} A^i & i < n \\ \ker d_A^i & i = n \\ 0 & i > n \end{cases}$$

Diagrammatically it looks like

$$\cdots \to A^{i-2} \xrightarrow{d_A^{i-2}} A^{i-1} \xrightarrow{\widehat{d_A^{i-1}}} \ker d_A^i \to 0 \to \cdots$$

The morphism $\widehat{d_A^{i-1}}$ is induced by the universal property of the kernel using $d_A^i d_A^{i-1} = 0$. There is a canonical momorphism $\iota^\bullet : \tau_{\leq n}(A^\bullet) \to A^\bullet$. This is the identity in degrees $< n$, zero in degrees $> n$, and the canonical morphism κ associated to the kernel in degree n.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^{i-2} & \xrightarrow{d_A^{i-2}} & A^{i-1} & \xrightarrow{\widehat{d_A^{i-1}}} & \ker d_A^i & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & A^{i-2} & \xrightarrow[d_A^{i-2}]{} & A^{i-1} & \xrightarrow[d_A^{i-1}]{} & A^i & \xrightarrow[d_A^i]{} & A^{i+1} & \longrightarrow & \cdots \end{array}$$

Furthermore, $\iota^\bullet$ induces isomorphisms on cohomology for degrees $\leq n$.

$$H^i(\iota^\bullet) : H^i(\tau_{\leq n}(A^\bullet)) \xrightarrow{\cong} H^i(A^\bullet) \qquad \forall i \leq n$$

If $f^\bullet : A^\bullet \to B^\bullet$ is a chain map, then as previously discussed we get an induced map $f_0^n : \ker d_A^n \to \ker d_B^n$ which we use to obtain a chain map $\tau_{\leq n}(f^\bullet) : \tau_{\leq n}(A^\bullet) \to \tau_{\leq n}(B^\bullet)$.

$$\tau_{\leq n}(f^\bullet) = \begin{cases} f^i & i < n \\ f_0^i & i = n \\ 0 & i > n \end{cases}$$

Hence $\tau_{\leq n}$ gives a covariant functor $C(\mathcal{A}) \to C(\mathcal{A})$. Even better, the image lands in the category $C^-(\mathcal{A})$ of complexes bounded above.

$$\tau_{\leq n} : C(\mathcal{A}) \to C^-(\mathcal{A})$$

Definition 2.59. The **left truncation of $A^\bullet$ at n** is the complex $\tau_{\geq n}(A^\bullet)$ defined by

$$\tau_{\geq n}(A^\bullet)^i = \begin{cases} 0 & i < n \\ \operatorname{coker} d_A^{i-1} & i = n \\ A^i & i > n \end{cases}$$

Diagrammatically it looks like

$$\cdots \to 0 \to \operatorname{coker} d_A^{i-1} \xrightarrow{\widehat{d_A^i}} A^{i+1} \xrightarrow{d_A^{i+1}} A^{i+2} \to \cdots$$

The morphism $\widehat{d_A^i}$ is induced by the universal property of the cokernel using $d_A^i d_A^{i-1} = 0$. There is a canonical epimorphism $q^\bullet : A^\bullet \to \tau_{\geq n}(A^\bullet)$. This is zero in degrees $< n$, the identity in degrees $> n$, and the canonical morphism ρ associated to the cokernel in degree n.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \xrightarrow{d_A^{i+1}} & A^{i+2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} d_A^{i-1} & \xrightarrow[\widehat{d_A^i}]{} & A^{i+1} & \xrightarrow[d_A^{i+1}]{} & A^{i+2} & \longrightarrow & \cdots \end{array}$$

Furthermore, $q^\bullet$ induces isomorphisms on cohomology for degrees $\geq n$.

$$H^i(q^\bullet) : H^i(A^\bullet) \xrightarrow{\cong} H^i(\tau_{\geq n}(A^\bullet)) \qquad \forall i \geq n$$

If $f^\bullet : A^\bullet \to B^\bullet$ is a chain map, then as previously discussed we get an induced map $f_1^n : \operatorname{coker} d_A^{n-1} \to \operatorname{coker} d_B^{n-1}$ which we use to obtain a chain map $\tau_{\geq n}(f^\bullet) : \tau_{\geq n}(A^\bullet) \to \tau_{\geq n}(B^\bullet)$.

$$\tau_{\geq n}(f^\bullet) = \begin{cases} 0 & i < n \\ f_1^i & i = n \\ f^i & i > n \end{cases}$$

Hence $\tau_{\geq n}$ gives a covariant functor $C(\mathcal{A}) \to C(\mathcal{A})$. Even better, the image lands in the category $C^+(\mathcal{A})$ of complexes bounded below.

$$\tau_{\leq n} : C(\mathcal{A}) \to C^+(\mathcal{A})$$

2.4.3 Cone of a morphism

As always, $\mathcal{A}$ is a fixed abelian category.

Definition 2.60. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$. The **cone** of $f^\bullet$ is a chain complex $C_f^\bullet = (C_f^n, d_{C_f}^n)$ which we now describe. The objects are

$$C_f^n := X[1]^n \oplus Y^n = X^{n+1} \oplus Y^n$$

and the differentials are

$$d^n_{C_f} : C^n_f \to C^{n+1}_f \qquad d_{C_f} = \begin{pmatrix} d_{X^\bullet[1]} & 0 \\ f^\bullet[1]^n & d^n_Y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$$

The matrix notation is slightly abusive, but it is at least clear what this means if $\mathcal{A}$ is a category of R-modules. In this case, we can write an element of $X^{n+1} \oplus Y^n$ as a "column vector" $\begin{pmatrix} x \\ y \end{pmatrix}$ and d_{C_f} acts on the left in the usual way matrices act on column vectors.

$$d_{C_f} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X(x) \\ f^{n+1}(x) + d^n_Y(y) \end{pmatrix}$$

This makes sense because f^{n+1} and d^n_Y both map into Y^{n+1}, where we can add their images (as Y^{n+1} is a module over a ring R). The matrix notation also works in a general abelian category, just be careful to understand things in terms of the categorical biproduct instead of in terms of elements.

A quick matrix calculation shows that $d^2_{C_f} = 0$, hence $C^\bullet_f$ is a chain complex as claimed.

$$\begin{pmatrix} -d^{n+2}_X & 0 \\ f^{n+2} & d^{n+1}_Y \end{pmatrix} \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} = \begin{pmatrix} d^{n+1}_X d^{n+2}_X & d^n_Y d^{n+1}_Y \\ -f^{n+2} d^{n+1}_X + d^{n+1}_Y f^{n+1} & d^{n+1}_Y d^n_Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The top row and bottom right are clearly zero since $X^\bullet, Y^\bullet$ are complexes. The bottom left entry vanishes because f is a chain map.

Definition 2.61. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a chain map, with cone $C^\bullet_f$. For $n \in \mathbb{Z}$, let

$$\iota^n_f = 0 \oplus \mathrm{Id}_{Y_n} : Y^n \to C^n_f$$

and let

$$\rho^n_f : C^n_f \to X[1]^n$$

be the canonical projection. Then one can verify that these give chain maps

$$\iota^\bullet_f : Y^\bullet \to C^\bullet_f$$
$$\rho^\bullet_f : C_f \to X^\bullet[1]$$

Remark 2.62. The chain maps $\iota^\bullet_f, \rho^\bullet_f$ fit into a short exact sequence of complexes

$$0 \to Y^\bullet \xrightarrow{\iota^\bullet_f} C^\bullet_f \xrightarrow{\rho^\bullet_f} X^\bullet[1] \to 0$$

which induces a long exact sequence on cohomology

$$\cdots \to H^n(C^\bullet_f) \to H^n(X^\bullet[1]) \xrightarrow{\partial^n} H^{n+1}(Y^\bullet) \to H^{n+1}(C^\bullet_f) \to \cdots$$

Recall that $H^n(X^\bullet[1]) = H^{n+1}(X)$. Tracing through the construction of the connecting map ∂^n in the snake lemma, we see that it is just the morphism induced on cohomology by f. That is,

$$\partial^n = H^{n+1}(f) : H^{n+1}(X^\bullet) \to H^{n+1}(Y^\bullet)$$

Proposition 2.63. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a chain map with cone $C_f^\bullet$. Then $f^\bullet$ is a quasi-isomorphism if and only if $C_f^\bullet$ is acyclic.*

Proof. Use the long exact sequence from the previous remark. $C_f^\bullet$ is acyclic if and only every third term vanishes, which is equivalent to all the connecting homomorphisms being isomorphisms. But these connecting homomorphisms are exactly the morphisms induced on cohomology by f. □

2.5 Derived functors - classical definition

2.5.1 Injective and projective objects

Fix an abelian category $\mathcal{A}$.

Definition 2.64. Recall that the contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, X)$ from $\mathcal{A}$ to abelian groups is left exact. An object I in $\mathcal{A}$ is **injective** if $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is exact.

Definition 2.65. Recall that the covariant functor $\operatorname{Hom}_{\mathcal{A}}(X, -)$ is right exact. An object P is **projective** if $\operatorname{Hom}_{\mathcal{A}}(P, -)$ is exact.

Remark 2.66. Suppose $\mathcal{A}$ is the category of modules over a commutative ring R, and let M be an R-module.

1. If R is a PID, then M is injective if and only if it is divisible.

2. M is projective if and only if it is a direct summand of a free module.

Definition 2.67. $\mathcal{A}$ has **enough injectives** if for any object X, there is an injective object I and a monomorphism $X \to I$.

Definition 2.68. $\mathcal{A}$ has **enough projectives** if for any object X, there is a projective object P and an epimorphism $P \to X$.

Example 2.69. The category of R-modules has enough projectives and enough injectives.

Definition 2.70. An **injective resolution** of an object X is an exact sequence

$$0 \to X \to I^0 \to I^1 \to \cdots$$

where I^n is injective for every n. We can also denote such a resolution by $0 \to X \to I^\bullet$.

Proposition 2.71. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

1. *Every object in $\mathcal{A}$ has an injective resolution.*

2. *Suppose $0 \to X \to M^\bullet$ is a long exact sequence and $0 \to Y \to I^\bullet$ is an injective resolution of Y. Then any morphism $f : X \to Y$ extends to a chain map*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & M^0 & \longrightarrow & M^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \vdots & & \vdots & & \\ 0 & \longrightarrow & Y & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

Moreover, any two such extensions are chain homotopic.

3. *(Horseshoe lemma) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathcal{A}$. Given injective resolutions $0 \to X \to I^\bullet$ and $0 \to Z \to K^\bullet$, there exists an injective resolution $0 \to Y \to J^\bullet$ of Y fitting into a short exact sequence of complexes as depicted below.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \downarrow & & \\
0 & \longrightarrow & I^0 & \cdots\to & J^0 & \cdots\to & K^0 & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \downarrow & & \\
0 & \longrightarrow & I^1 & \cdots\to & J^1 & \cdots\to & K^1 & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

In fact, one can take $J^n = I^n \oplus K^n$.

Proof. Omitted, proven in various introductory notes or texts on homological algebra. □

Remark 2.72. It follows from part 2 of the proposition that if $I_1^\bullet, I_2^\bullet$ are two injective resolutions of an object, then $0 \to A \to I_1^\bullet$ and $0 \to A \to I_2^\bullet$ are chain homotopy equivalent complexes. Just apply the extension property to the identity $A \to A$ both ways.

Remark 2.73. Part 3 of the previous proposition is known as the "Horseshoe lemma" for the shape of the diagram (before filling in the J terms, and omitting the zeros).

Remark 2.74. In part 3 of the proposition, the fact that $J^n = I^n \oplus K^n$ has the following consequence. If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then it commutes with direct sums, so it takes split exact sequences to split exact sequences. In particular, since

$$0 \to I^n \to J^n \to K^n \to 0$$

is split exact, the sequence

$$0 \to FI^n \to FJ^n \to FK^n \to 0$$

is also split exact. The split part is not so important, the emphasis here is that the resulting sequence is exact.

Remark 2.75. Let A be an object of $\mathcal{A}$, and let $0 \to A \xrightarrow{i_0} I^\bullet$ be an injective resolution of A. Then we have a morphism of complexes $C(A) \to I^\bullet$.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle i_0} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}$$

Even better, this is a quasi-isomorphism. This allows us to rephrase the fact that injective resolutions exist. This says that existence of an injective resolution of A is equivalent to the existence in $C(\mathcal{A})$ of a complex consisting of injective objects which is quasi-isomorphic to $C(A)$.

2.5.2 Derived functor construction

We now describe the classical (and somewhat ad hoc) approach to defining and calculating derived functors.

Definition 2.76. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and assume $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact (additive, covariant) functor. Given an object A of $\mathcal{A}$, choose an injective resolution $0 \to A \to I^\bullet$. Apply F to this resolution and truncate the A term. We obtain a complex

$$0 \to FI^0 \to FI^1 \to FI^2 \to \cdots$$

Then for $n \geq 0$, we define the n**th right derived functor of** F by

$$R^n F(A) := H^n(FI^\bullet)$$

Then $R^n F$ is an additive covariant functor $\mathcal{A} \to \mathcal{B}$.

Remark 2.77. The previous construction has the following important properties.

1. While the definition $R^n F(A) = H^n(FI^\bullet)$ appears to depend on the choice of injective resolution, it does not. This follows from 2.71. More precisely, it follows from remark 2.72, which says that injective resolutions of an object A are chain homotopy equivalent.

2. $R^0 F(A) = F(A)$ since F is left exact.

3. If I is injective, then $R^n F(I) = 0$ for $n \geq 1$, using the injective resolution $0 \to I \to I \to 0$.

4. A short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ yields a long exact sequence

$$0 \to R^0FA \to R^0FB \to R^0FC \to\to R^1FA \to R^1FB \to R^1FC \to R^2FA \to \cdots$$

   This comes from applying 2.54 to the short exact sequence of complexes obtained in part 3 of 2.71. An important step is that since F is additive, it preserves coproducts, and the obtained resolution of B is the term-by-term coproduct.

Remark 2.78. All of this dualizes to projective resolutions and left derived functors, but we omit the details.

2.6 Homotopy category of complexes

2.6.1 Definitions

Next we define the homotopy category $K(\mathcal{A})$. It will have the same objects as $C(\mathcal{A})$, but the hom sets are quotients of the $C(\mathcal{A})$ hom sets.

Definition 2.79. A chain map is **nullhomotopic** if it is chain homotopic to the zero chain map. Explicitly, $f^\bullet : A^\bullet \to B^\bullet$ is nullhomotopic if there exist maps $h^n : A^n \to B^{n-1}$ such that

$$f^n = h^{n+1} d_A^n + d_B^{n-1} h^n$$

Note that $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are chain homotopic if and only if $f^\bullet - g^\bullet$ is nullhomotopic.

Definition 2.80. Let $\mathcal{A}$ be an abelian category. The **homotopy category of complexes**, denoted $K(\mathcal{A})$, is the category whose objects are the same as objects of $C(\mathcal{A})$, and whose morphism are given by

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)/G$$

where G is the subgroup of nullhomotopic chain maps.

We haven't yet established that the nullhomotopic chain maps form a subgroup, so the lemma takes care of this, and shows that composition in $K(\mathcal{A})$ is well defined.

Lemma 2.81. *Let $\mathcal{A}$ be an abelian category.*

1. *Let $f^\bullet : X^\bullet \to Y^\bullet$ and $g^\bullet : Y^\bullet \to Z^\bullet$ be chain maps. If $f^\bullet$ or $g^\bullet$ is nullhomotopic, then so is $g^\bullet f^\bullet$.*
2. *If $f^\bullet : X^\bullet \to Y^\bullet$ is nullhomotopic, then $H^n(f^\bullet) = 0$ for all n. Consequently, any chain homotopy equivalence is a quasi-isomorphism.*

Proof. (1) Suppose f is nullhomotopic. Then there is $h^n : X^n \to Y^{n-}1$ such that

$$f^n = h^{n+1} d_X^n + d_Y^{n-1} h^n$$

Then

$$g^n f^n = g^n h^{n+1} d_X^n + g^n d_Y^{n-1} h^n = g^n h^{n+1} d_X^n + d_Z^{n-1} g^{n-1} h^n$$

so $g^{n-1} h^n$ gives a chain homotopy between $g^\bullet f^\bullet$ and the zero morphism. A similar argument shows that if $g^\bullet$ is nullhomotopic, then $g^\bullet f^\bullet$ is nullhomotopic.

(2) Apply the Freyd-Mitchell embedding theorem and use the usual argument for R-modules. □

Remark 2.82. Part (1) of the lemma says that composition is the homotopy category $K(\mathcal{A})$ is well defined, since the choice of homotopy class representative doesn't impact composition.

Remark 2.83. $K(\mathcal{A})$ is not, in general, an abelian category. We will prove this later, with examples. The remedy/approximation for this will be that $K(\mathcal{A})$ is a triangulated category.

2.6.2 Translation and truncation

Lemma 2.84. *If $f^\bullet : X^\bullet \to Y^\bullet$ is nullhomotopic, then so is $T(f^\bullet) : T(X^\bullet) \to T(Y^\bullet)$. More generally, if $f^\bullet, g^\bullet$ are chain homotopic, then so are $T(f^\bullet), T(g^\bullet)$.*

Proof. Just translate the homotopy maps by one degree. □

Remark 2.85. The previous remark says that the translation functor $T : C(\mathcal{A}) \to C(\mathcal{A})$ induces a translation functor $T : K(\mathcal{A}) \to K(\mathcal{A})$.

Definition 2.86. The homotopy category has bounded subcategories $K^+(\mathcal{A}), K^-(\mathcal{A}), K^b(\mathcal{A})$ which are defined analogously to $C^+(\mathcal{A}), C^-(\mathcal{A}), C^b(\mathcal{A})$.

Lemma 2.87. *Let $n \in \mathbb{Z}$. If $f^\bullet : X^\bullet \to Y^\bullet$ is nullhomotopic, then $\tau_{\leq n}(f^\bullet)$ is also nullhomotopic.*

Proof. Let $\kappa_X^n : \ker d_X^n \to X^n, \kappa_Y^n : \ker d_Y^n \to Y^n$ be the canonical maps associated with the kernel. The chain map $\tau_{\leq n}(f^\bullet)$ in $C(\mathcal{A})$ looks like this.

$$\begin{CD} \cdots @>>> X^{n-2} @>>> X^{n-1} @>{\widehat{d}_X^{n-1}}>> \ker d_X^n @>>> 0 @>>> \cdots \\ @. @VV{f^{n-2}}V @VV{f^{n-1}}V @VV{f_0^n}V @VV{0}V \\ \cdots @>>> Y^{n-2} @>>> Y^{n-1} @>{\widehat{d}_Y^{n-1}}>> \ker d_Y^n @>>> 0 @>>> \cdots \end{CD}$$

where $\kappa_Y^n f_0^n = f^n \kappa_X^n$, and $\kappa_Y^n \widehat{d}_Y^{n-1} = d_Y^{n-1}$.

$$\begin{CD} \ker d_X^n @>{\kappa_X^n}>> X^n \\ @VV{f_0^n}V @VV{f^n}V \\ \ker d_Y^n @>{\kappa_Y^n}>> Y^n \end{CD} \qquad \begin{array}{ccc} & & Y^{n-1} \\ & \overset{\widehat{d}_Y^{n-1}}{\swarrow} & \big\downarrow{\scriptstyle d_Y^{n-1}} \\ \ker d_Y^n & \xrightarrow{\kappa_Y^n} & Y^n \end{array}$$

Since $f^\bullet$ is nullhomotopic, there exist maps $h^k : X^k \to Y^{k-1}$ such that

$$f^k = h^{k+1} d_X^k + d_Y^{k-1} h^k \qquad \forall k \in \mathbb{Z}$$

We define

$$h_\tau^k = \begin{cases} h^k & k < n \\ h^n \kappa_X^n & k = n \\ 0 & k > n \end{cases}$$

You can think of the composition $h^n \kappa_X^n$ as a "restriction" $h^n|_{\ker d_X^n}$. We claim that the maps h_τ^k give a nullhomotopy of $\tau_{\leq n}(f^\bullet)$. This is immediate in degrees other than n, so we only need to check it in degree n.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & \ker d_X^n & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \big\downarrow & \overset{h^{n-1}}{\swarrow} & \big\downarrow & \overset{h^n\kappa_X^n}{\swarrow} & \big\downarrow & \overset{0}{\swarrow} & \big\downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & Y^{n-2} & \longrightarrow & Y^{n-1} & \longrightarrow & \ker d_Y^n & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Then we need to verify

$$f_0^n \stackrel{?}{=} h_\tau^{n+1} \circ 0 + \widehat{d}_Y^{n-1} \circ h_\tau^n$$

We start by composing f_0^n with κ_Y^n on the left, then use our various properties.

$$\begin{aligned}
\kappa_Y^n f_0^n &= f^n \kappa_X^n && \text{construction of } f_0^n \\
&= \left(h^{n+1} d_X^n + d_Y^{n-1} h^n\right) \kappa_X^n && \text{nullhomotopy of } f \\
&= h^{n+1} d_X^n \kappa_X^n + d_Y^{n-1} h^n \kappa_X^n && \text{distributivity} \\
&= d_Y^{n-1} h^n \kappa_X^n && d_X^n \kappa_X^n = 0 \text{ by definition of kernel} \\
&= \kappa_Y^n \widehat{d}_Y^{n-1} h^n \kappa_X^n && \text{construction of } \widehat{d}_Y^{n-1}
\end{aligned}$$

Finally, since κ_Y^n is a monomorphism, we can left cancel it, and obtain the needed equality.

$$f_0^n = \widehat{d}_Y^{n-1} h^n \kappa_X^n = h_\tau^{n+1} \circ 0 + \widehat{d}_Y^{n-1} \circ h_\tau^n$$

□

3 Triangulated categories

In some ways, being a triangulated category is a "weaker" requirement than being abelian, but really, they are just orthogonal. Neither abelian nor triangulated implies the other, and eventually we can show that having both properties is extremely restrictive, so in some sense "most" abelian categories are not triangulated, and "most" triangulated categories are not abelian.

Triangulated categories take a different approach than abelian categories to having something akin to short exact sequences. Instead of kernels and cokernels and strict morphisms, triangulated categories bypass these notions and start out with "distinguished triangles," which function in many ways like exact sequences. Since there are not necessarily kernels and cokernels, we cannot speak of exactness per se, so this is the replacement.

Informally, a triangulated category is an additive category with a translation functor and a family of distinguished triangles satisfying some axioms.

3.1 Axioms for a triangulated category

Definition 3.1. Let $\mathcal{C}$ be an additive category. A **translation functor** on $\mathcal{C}$ is an automorphism[2] $T : \mathcal{C} \to \mathcal{C}$. When we have such a T, we notate things as

$$T^n(X) = X[n] \qquad n \in \mathbb{Z}$$

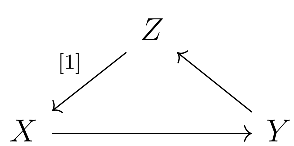
and if $f : X \to Y$ is a morphism,

$$T^n(f) = f[n] : X[n] \to Y[n]$$

Definition 3.2. Let $\mathcal{C}$ be a category with a translation functor. A **triangle** in $\mathcal{C}$ is a diagram

$$X \to Y \to Z \to X[1]$$

Note that there are no assumptions regarding "exactness," since this would not even have a clear meaning. A triangle is sometimes represented



Such a diagram can be misleading if taken too literally - there is no actual morphism $Z \to X$, which is why the arrow is decorated with a [1]. This label is not a name of a morphism, it indicates a morphism $Z \to X[1]$.

Definition 3.3. Let $\mathcal{C}$ be a category with a translation functor. A **morphism of triangles** is a commutative diagram

[2]Automorphism is much stronger than an equivalence of categories with itself. An automorphism means there is an inverse functor $T^{-1} : \mathcal{C} \to \mathcal{C}$ such that TT^{-1} and $T^{-1}T$ are both the identity functor on $\mathcal{C}$.

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

A morphism of triangles is an **isomorphism of triangles** if u, v, w are all isomorphisms.

Definition 3.4. A **triangulated category** is an additive category $\mathcal{C}$ which is equipped with a translation functor $T : \mathcal{C} \to \mathcal{C}$ (also denoted [1]) and a family of triangles called **distinguished triangles** satisfying the following axioms.

(TR1a) For any object X, the triangle $X \xrightarrow{\text{Id}} X \to 0 \to X[1]$ is distinguished.

(TR1b) Any triangle isomorphic to a distinguished triangle is distinguished.

(TR1c) Any morphism $X \xrightarrow{f} Y$ can be complete to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

The resulting object Z is sometimes called the **cone** of f. [3]

(TR2) (Rotation axiom) A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is distinguished if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is distinguished. (This second triangle is called the **rotated triangle**.)

(TR3) Suppose we have the following diagram with distinguished rows, and the left square commutes.

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

Then this can be completed to a morphism of triangles. That is, there exists a morphism $h : Z \to Z'$ (not necessarily unique) making the following diagram commute.

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

[3]Note that we do not assume Z is unique, but can eventually show that Z is unique up to isomorphism, see Remark 3.20. However, Z is not unique up to unique isomorphism, so there is no canonical choice of Z, nor is there a canonical choice of isomorphism $Z \cong Z'$ given two such completions.

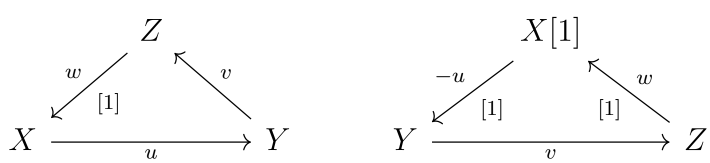
(TR4) (Octahedral axiom) Suppose we have the following diagram with distinguised rows, and the two squares on the left commute ($gf = h$).

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{a} & Z' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle \mathrm{Id}_X} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \mathrm{Id}_X[1]=\mathrm{Id}_{X[1]}} \\
X & \xrightarrow{h} & Z & \xrightarrow{b} & Y' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}_Z} & & & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \xrightarrow{c} & X' & \longrightarrow & Y[1]
\end{array}$$

Then this can be completed to two morphisms of triangles. That is, there exist morphisms $u : Z' \to Y'$ and $v : Y' \to X'$ and $w : X' \to Z'[1]$ making the whole diagram commute, so that the bottom row is also distinguished.

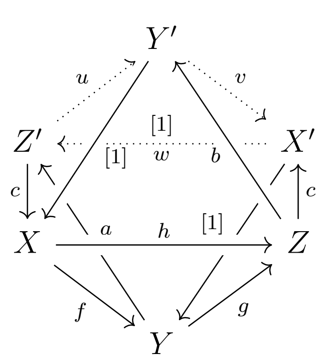
$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{a} & Z' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle \mathrm{Id}_X} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{Id}_X[1]=\mathrm{Id}_{X[1]}} \\
X & \xrightarrow{h} & Z & \xrightarrow{b} & Y' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}_Z} & & \vdots{\scriptstyle v} & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \xrightarrow{c} & X' & \longrightarrow & Y[1] \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \mathrm{Id}_{X'}} & & \downarrow{\scriptstyle a[1]} \\
Z' & \overset{u}{\dashrightarrow} & Y' & \overset{v}{\dashrightarrow} & X' & \overset{w}{\dashrightarrow} & Z'[1]
\end{array}$$

Remark 3.5. The name "rotation axiom" for (TR2) comes from the following picture.



The rotation axiom is telling us that we can take a distinguished triangle as on the left and "rotate" it counterclockwise to obtain a new distinguished triangle, on the right. Since the statement is if and only if, it also includes the case of rotating clockwise.

Remark 3.6. The name "octahedral axiom" for (TR4) comes from the following picture. If you understand what's going on in this picture, then you're smarter than I am.



Definition 3.7. A category with a translation functor and family of distinguished triangles satisfying axioms (TR1a), (TR1b), (TR1c), (TR2), (TR3), is called **pre-triangulated**.

Remark 3.8. It is an open research question whether there exists a category which is pre-triangulated but not triangulated. That is to say, we do not know if (TR4) is a consequence of the other axioms. The general consensus seems to be that it is not, and to prove this, one would just need to find a pre-triangulated category in which (TR4) fails. However, no such category is known.

Definition 3.9. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories, with translation functors $T_\mathcal{C}, T_\mathcal{D}$, and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. F **commutes with translation** if there is a natural isomorphism of functors $\eta : FT_\mathcal{C} \xrightarrow{\cong} T_\mathcal{D}F$. That is, for every object X of $\mathcal{C}$, there is an isomorphism

$$\eta_X : FT_\mathcal{C}X \xrightarrow{\cong} T_\mathcal{D}FX$$

and if $f : X \to Y$ is a morphism in $\mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccc} FT_\mathcal{C}X & \xrightarrow[\cong]{\eta_X} & T_\mathcal{D}FX \\ {\scriptstyle FT_\mathcal{C}f}\downarrow & & \downarrow{\scriptstyle T_\mathcal{D}Ff} \\ FT_\mathcal{C}Y & \xrightarrow[\cong]{\eta_Y} & T_\mathcal{D}FY \end{array}$$

If we use [1] to denote translation in both $\mathcal{C}, \mathcal{D}$, then we would notate this as

$$\begin{array}{ccc} F(X[1]) & \xrightarrow[\cong]{\eta_X} & (FX)[1] \\ {\scriptstyle F(f[1])}\downarrow & & \downarrow{\scriptstyle (Ff)[1]} \\ F(Y[1]) & \xrightarrow[\cong]{\eta_Y} & (FY)[1] \end{array}$$

Definition 3.10. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. F is **triangulated** or **exact** if it commutes with translation and takes distinguished triangles to distinguished triangles. That is, if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle in $\mathcal{C}$, then

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\eta_X Fw} (FX)[1]$$

is a distinguished triangle in $\mathcal{D}$.

Note that without the natural isomorphism η, the resulting sequence after applying F would not even be a triangle, so it wouldn't make any sense to talk about it being distinguished. Without η the image of $X \to Y \to Z \to X[1]$ would be $FX \to FY \to FZ \to F(X[1])$, and then there's no reason to expect any relationship between $F(X[1])$ and $(FX)[1]$, except that we assumed F commutes with translation.

Definition 3.11. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be exact functors. Let $\eta_F : FT_\mathcal{C} \to T_\mathcal{D}F$ and $\eta_G : GT_\mathcal{C} \to T_\mathcal{D}G$ be the associated natural isomorphisms from the fact that F, G each commute with translation. A natural transformation $\omega : F \to G$ is **graded** if the following diagram commutes for every $X \in \text{ob}(\mathcal{C})$.

$$\begin{array}{ccc} FT_{\mathcal{C}}X & \xrightarrow[\cong]{\eta_{F,X}} & T_{\mathcal{D}}FX \\ {\scriptstyle \omega_{T_{\mathcal{C}}X}}\downarrow & & \downarrow{\scriptstyle T_{\mathcal{D}}\omega_X} \\ GT_{\mathcal{C}}X & \xrightarrow[\cong]{\eta_{G,X}} & T_{\mathcal{D}}GX \end{array}$$

Remark 3.12. If $\omega : F \to G$ is a graded natural transformation as above, then for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{C}$, there is a commutative diagram

$$\begin{array}{ccccccccc} FX & \longrightarrow & FY & \longrightarrow & FZ & \longrightarrow & FT_{\mathcal{C}}X & \xrightarrow{\eta_{F,X}} & T_{\mathcal{D}}FX \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_Y} & & \downarrow{\scriptstyle \omega_Z} & & \downarrow{\scriptstyle \omega_{T_{\mathcal{C}}X}} & & \downarrow{\scriptstyle T_{\mathcal{D}}\omega_X} \\ GX & \longrightarrow & GY & \longrightarrow & GZ & \longrightarrow & GT_{\mathcal{C}}X & \longrightarrow & T_{\mathcal{D}}GX \end{array}$$

If we collapse the rightmost square, this becomes a morphism of distinguished triangles.

$$\begin{array}{ccccccc} FX & \longrightarrow & FY & \longrightarrow & FZ & \longrightarrow & T_{\mathcal{D}}FX \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_Y} & & \downarrow{\scriptstyle \omega_Z} & & \downarrow{\scriptstyle T_{\mathcal{D}}\omega_X} \\ GX & \longrightarrow & GY & \longrightarrow & GZ & \longrightarrow & T_{\mathcal{D}}GX \end{array}$$

3.2 Some properties of triangulated categories

Proposition 3.13. *If* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is a distinguished triangle, then* $vu = 0$ *and* $wv = 0$.

Proof. It is enough to prove that $vu = 0$, since then $wv = 0$ follows by applying the rotation axiom (TR2). We know that $Z \xrightarrow{\mathrm{Id}} Z \to 0 \to Z[1]$ is distinguished by (TR1a). And we have the following diagram with distinguished rows, and the left square is clearly commutative.

$$\begin{array}{ccccccc} Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \mathrm{Id}} & & & & \downarrow{\scriptstyle v[1]} \\ Z & \xrightarrow{\mathrm{Id}} & Z & \longrightarrow & 0 & \longrightarrow & Z[1] \end{array}$$

Thus by (TR3), there is a morphism $X[1] \to 0$ which completes this to a morphism of triangles. Obviously, the only morphism it could be is the zero morphism.

$$\begin{array}{ccccccc} Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow{\scriptstyle v[1]} \\ Z & \xrightarrow{\mathrm{Id}} & Z & \longrightarrow & 0 & \longrightarrow & Z[1] \end{array}$$

But we still get useful information, since it tells us that the right square commutes, that is,

$$v[1] \circ -u[1] = 0 \implies vu = 0$$

since [1] is an automorphism. $\square$

Proposition 3.14. *If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is a distinguished triangle, then any change of sign for exactly two of u, v, w is still a distinguished triangle.*

Proof. This is immediate from (TR1b) using a diagram such as the following.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle -\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ X & \xrightarrow{-u} & Y & \xrightarrow{-v} & Z & \xrightarrow{w} & X[1] \end{array}$$

□

Definition 3.15. Let $\mathcal{C}$ be any category (not even necessarily additive). Let $f : A \to B$ be a morphism and U any object. We denote the induced maps on Hom-sets by f_* and f^* respectively.

$$\begin{array}{ll} f_* : \mathrm{Hom}_{\mathcal{C}}(U, A) \to \mathrm{Hom}_{\mathcal{C}}(U, B) & \qquad f_*\phi = f\phi = f \circ \phi \\ f^* : \mathrm{Hom}_{\mathcal{C}}(B, U) \to \mathrm{Hom}_{\mathcal{C}}(A, U) & \qquad f^*\psi = \psi f = \psi \circ f \end{array}$$

The next result gives a strong backing to the philosophical idea that distinguished triangles behave a lot like short exact sequences of complexes - they induce long exact sequences of abelian groups in a similar way to how the snake lemma is used to get a long exact sequence from a short exact sequence of complexes.

Proposition 3.16. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a distinguished triangle in a triangulated category $\mathcal{C}$, and let U be any object of $\mathcal{C}$. Then the following are long exact sequences of abelian groups.*

$$\cdots \longrightarrow \mathrm{Hom}(U, X[i]) \xrightarrow{u[i]_*} \mathrm{Hom}(U, Y[i]) \xrightarrow{v[i]_*} \mathrm{Hom}(U, Z[i]) \xrightarrow{w[i]_*} \mathrm{Hom}(U, X[i+1]) \longrightarrow \cdots$$
$$\cdots \longleftarrow \mathrm{Hom}(X[i], U) \xleftarrow{u[i]^*} \mathrm{Hom}(Y[i], U) \xleftarrow{v[i]_*} \mathrm{Hom}(Z[i], U) \xleftarrow{w[i]_*} \mathrm{Hom}(X[i+1], U) \longleftarrow \cdots$$

Proof. We'll just prove exactness for the first sequence, since the proof for the other is analogous. Because of the rotation axiom (TR2), it suffices to prove exactness at a single term. So we just need to show that

$$\mathrm{Hom}(U, X[i]) \xrightarrow{u[i]_*} \mathrm{Hom}(U, Y[i]) \xrightarrow{v[i]_*} \mathrm{Hom}(U, Z[i])$$

is exact at the middle term. By proposition 3.13, $vu = 0$, so $v[i] \circ u[i] = 0$ so $\mathrm{im}\, u[i]_* \subset \ker v[i]_*$. We just need to establish the reverse inclusion. Suppose $f \in \ker v[i]_*$, so $f \circ v[i] = 0$. We need to find a $g : U \to X[i]$ such that $f = u[i]_* g = u[i] \circ g$. Using (TR1a) and (TR2) we have the following diagram with distinguished rows.

$$\begin{array}{ccccccc} U[-i] & \longrightarrow & 0 & \longrightarrow & U[-i+1] & \xrightarrow{-\mathrm{Id}} & U[-i+1] \\ \downarrow{\scriptstyle f[-i]} & & \downarrow & & & & \downarrow{\scriptstyle f[-i+1]} \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \end{array}$$

The left square clearly commutes, so by (TR3) there is a morphism h completing this to a morphism of triangles.

$$\begin{array}{ccccccc} U[-i] & \longrightarrow & 0 & \longrightarrow & U[-i+1] & \xrightarrow{-\mathrm{Id}} & U[-i+1] \\ \downarrow{\scriptstyle f[-i]} & & \downarrow & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[-i+1]} \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \end{array}$$

That is,

$$-f[-i+1] = -u[1] \circ h \implies f = u[i] \circ h[-i+1]$$

so we can take $g = h[-i+1]$, and then $f = u[i]_*(h)$ so $f \in \operatorname{im} u[i]_*$. Thus the sequence is exact. ☐

Remark 3.17. The proof of proposition 3.16 did not utilize (TR4), so it holds in pre-triangulated categories as well. This will be true of most of the following results as well, so we'll stop mentioning it.

Corollary 3.18 (Triangulated 5-lemma)**.** *Suppose we have a morphism of distinguished triangles.*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

If two of f, g, h are isomorphisms, then the third is also.

Proof. Because of the rotation axiom (TR2), it suffices to prove the case where f, g are isomorphisms. We apply proposition 3.16 in the case $U = Z'$ to both rows, and get two long exact sequences, which are connected by vertical maps induced by f, g, h.

$$\begin{array}{ccccccccc} \mathrm{Hom}(Z', X) & \longrightarrow & \mathrm{Hom}(Z', Y) & \longrightarrow & \mathrm{Hom}(Z', Z) & \longrightarrow & \mathrm{Hom}(Z', X[1]) & \longrightarrow & \mathrm{Hom}(Z', Y[1]) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle f[1]_*} & & \downarrow{\scriptstyle g[1]_*} \\ \mathrm{Hom}(Z', X') & \longrightarrow & \mathrm{Hom}(Z', Y') & \longrightarrow & \mathrm{Hom}(Z', Z') & \longrightarrow & \mathrm{Hom}(Z', X'[1]) & \longrightarrow & \mathrm{Hom}(Z', Y'[1]) \end{array}$$

Note that this diagram is commutative simply because $\mathrm{Hom}(Z', -)$ is a functor. The rows are exact by 3.16. By assumption, f, g are isomorphisms, so $f_*, g_*, f[1]_*, g[1]_*$ are also isomorphisms. So by the 5-lemma, h_* is an isomorphism. Thus there is a map $\alpha : Z' \to Z$ such that $h_*(\alpha) = h\alpha = \mathrm{Id}_{Z'}$, that is, α is a right inverse for h.

We omit the details, but using the same diagrammatic argument using $U = Z$ instead, one can obtain a left inverse β for h. Then it follows formally that $\alpha = \beta$ is a two-sided inverse for h, so h is an isomorphism. ☐

Remark 3.19. The proof for the triangulated 5-lemma above does not depend on the octahedral axiom, so it holds in pre-triangulated categories as well.

Remark 3.20. By (TR1c), any morphism $u : X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$. There is no assumption of uniqueness for Z, but we can now establish that Z is actually unique up to non-canonical isomorphism. More precisely, suppose $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and $X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} X[1]$ are both completions from (TR1c). Then by (TR3), there is a morphism $h : Z \to Z'$ making a morphism of triangles.

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle \mathrm{Id}_X} & & \downarrow{\scriptstyle \mathrm{Id}_Y} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{Id}_X[1] = \mathrm{Id}_{X[1]}} \\ X & \longrightarrow & Y & \longrightarrow & Z' & \longrightarrow & X[1] \end{array}$$

Then by Corollary 3.18, h is an isomorphism. So it is somewhat reasonable to speak of "the cone" of u as the (isomorphism class of the) object Z. Despite this, there is no canonical representative for Z, and the morphisms v, w are only determined up to automorphisms of Z, and there is no canonical choice of h.

Corollary 3.21. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a distinguished triangle. Then u is an isomorphism if and only if $Z = 0$.*

Proof. We know that $vu = 0$, so we have a commutative diagram below with distinguished rows.

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{Id}} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \end{array}$$

If $Z = 0$, then the third vertical arrow is an isomorphism, which implies that u is an isomorphism by 3.18. Conversely, if u is an isomorphism, then $0 \to Z$ is an isomorphism again by 3.18. □

Definition 3.22. Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive covariant functor $H : \mathcal{C} \to \mathcal{A}$ is **cohomological** if for every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{C}$, the sequence

$$HX \xrightarrow{Hu} HY \xrightarrow{Hv} HZ$$

is exact in $\mathcal{A}$.

Lemma 3.23. *Let $H : \mathcal{C} \to \mathcal{A}$ be a cohomological functor. Then for every distinguished triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

the sequence below is exact.

$$\cdots \to H(X[i]) \xrightarrow{H(u[i])} H(Y[i]) \xrightarrow{H(v[i])} H(Z[i]) \xrightarrow{H(w[1])} H(X[i+1]) \to \cdots$$

Proof. Immediate from the definition combined with the rotation axiom (TR2). ☐

Example 3.24. We showed in proposition 3.16 that $\mathrm{Hom}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathrm{AbGp}$ is cohomological for any triangulated category $\mathcal{C}$ and any object U.

Remark 3.25. Later after we show $K(\mathcal{A})$ is triangulated, we'll show that $H^0 : C(\mathcal{A}) \to \mathcal{A}$ induces a functor $H^0 : K(\mathcal{A}) \to \mathcal{A}$ which is cohomological.

3.3 Semisimplicity

3.3.1 Semisimple rings and modules

Before we define semisimple abelian categories, we motivate the definition with a discussion of semisimple rings and semisimple modules. Mild warning: Zorn's lemma is used several times in the proofs in this section. If that's scary or you have some sort of philosophical objection, I don't know what to tell you. Deal with it.

Definition 3.26. Let A be an associative unital ring. A left A-module M is **simple** if it is nonzero and has no submodules other than 0 and itself.

Definition 3.27. Let A be an associative unital ring. A left A-module M is **semisimple** if for every submodule $N \subset M$, there exists a submodule $N' \subset M$ such that $M = N \oplus N'$. We call such an N' an **orthogonal complement** of N.

Remark 3.28. A simple module is semisimple (in a vacuous way).

Lemma 3.29. *Let A be an associative unital ring and let M be a semisimple left A-module. Then*

1. *Every submodule and quotient and quotient of M is semisimple.*
2. *If $M \neq 0$, then M contains a nonzero simple submodule.*
3. *Every $x \in M$ is contained in a nonzero simple submodule of M.*

Proof. (1) Let $L \subset M$ be a submodule, and let $N \subset L$ be a submodule of L. Since M is semisimple, there is a submodule $N' \subset M$ so that $N \oplus N' = M$. Then take $\widehat{N}' = N' \cap L$, and

$$N \oplus \widehat{N}' = N \oplus (N' \cap L) = (N \oplus N') \cap (N \cap L) = M \cap L = L$$

so L is semisimple. Regarding quotients, we know that there is a submodule L so that $M = L \oplus L'$, so

$$M/L \cong L'$$

but $L' \subset M$ is a submodule, so by the previous argument L' is semisimple. Hence the quotient M/L is semisimple.

(2) Since $M \neq 0$, there exists $x \in M, x \neq 0$. Consider the family of nonzero proper submodules

$$S = \{N \subset M : N \neq M, x \notin N\}$$

Since M is not simple, S is not empty. S is partially ordered by inclusion, and we claim that every chain in S has an upper bound. Let

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

be a chain in S. Then

$$\bigcup_i N_i$$

is a submodule of M which does not contain x, so it in S. Thus by Zorn's lemma, S has a maximal element $\widetilde{N}$. Since M is semisimple, there is a submodule which is an orthogonal complement to $\widetilde{N}$, that is, there is $\widetilde{N}'$ so that $M = \widetilde{N} \oplus \widetilde{N}'$. We claim $\widetilde{N}'$ is simple. Since $x \in \widetilde{N}$ and $\widetilde{N}$ is a proper submodule, $\widetilde{N}'$ is nonzero.

Suppose $\widetilde{N}'$ contains a nonzero proper submodule $L \subset \widetilde{N}'$. Since $\widetilde{N}'$ is semisimple there exists L' so that $\widetilde{N}' = L \oplus L'$. Then we have

$$\widetilde{N} \subsetneq \widetilde{N} \oplus L \qquad \widetilde{N} \subsetneq \widetilde{N} \oplus L'$$

so by maximality of $\widetilde{N}'$ in S, we know $x \in \widetilde{N} \oplus L$ and $x \in \widetilde{N} \subseteq N \oplus L'$. But then

$$x \in (\widetilde{N} \oplus L) \cap (\widetilde{N} \oplus L') = \widetilde{N}$$

which is a contradiction since $\widetilde{N} \in S$. □

Theorem 3.30. *Let A be an associative unital ring and M a left A-module. The following are equivalent.*

1. *M is semisimple.*
2. *M is a direct sum of simple modules.*
3. *M is a sum of simple modules.*

Proof. We'll just do a partial proof, just (1) $\iff$ (3).

(1) $\implies$ (3) Let $N \subset M$ be the sum of all simple submodules, and suppose $N \neq M$. So there is an N' so that $M = N \oplus N'$. Then N' is semisimple, so it contains a simple submodule, but then this simple submodule is not contained in N, which contradicts the definition of N.

(3) $\implies$ (1) Suppose $M = \sum_{i \in I} M_i$ with M_i simple submodules. Let $N \subset M$ be a proper submodule. Consider the family

$$S = \left\{ \text{subsets } K \subset I \text{ such that } N \cap \sum_{i \in K} M_i = 0 \right\}$$

Since $N \neq M$, $M_i \subsetneq N$ for at least some i, so S is nonempty. If we have an ascending chain in S, we can take the union of that ascending chain and obtain a new element of S, so S satisfies the hypotheses of Zorn's lemma. So by Zorn's lemma, S has a maximal element $\widetilde{K}$. Set

$$N' = \sum_{i \in K'} M_i$$

We claim that $M = N \oplus N'$. By definition of S, $N \cap N' = 0$, so it suffices to show that $M_i \subset N + N'$ for all $i \in I$. Suppose $M_i \not\subset N + N'$ for some i. Then $M_i \cap (N + N')$ is a proper submodule of M_i, so it is zero. Then

$$N \cap \sum_{j \in K' \cup \{i\}} M_j = 0$$

but this contradicts the maximality of K'. □

Example 3.31. Let $A = \mathbb{Z}$. We describe all the simple modules and some of the semisimple modules for A.

We claim that the simple $\mathbb{Z}$-modules are of the form $\mathbb{Z}/p\mathbb{Z}$ for p a prime. Suppose M is a simple $\mathbb{Z}$-module, and let $m \in M$. Then we have a $\mathbb{Z}$-module homomorphism (abelian group homomorphism)

$$\phi_m : \mathbb{Z} \to M \qquad 1 \mapsto m$$

Since M is simple, and $\operatorname{im} \phi_m$ is a nonzero submodule, ϕ_m must be surjective. Thus by the 1st isomorphism theorem $M \cong \mathbb{Z}/\ker \phi_m \cong \mathbb{Z}/n\mathbb{Z}$ for some n. If n is not prime, then $\mathbb{Z}/n\mathbb{Z}$ is not simple. If $n = ab$ with $a, b > 1$, then $a\mathbb{Z}/n\mathbb{Z}$ is a nonzero proper submodule of $\mathbb{Z}/n\mathbb{Z}$. On the other hand, if n is prime, then $\mathbb{Z}/n\mathbb{Z}$ is simple.

A semisimple $\mathbb{Z}$-module is a direct sum of simple modules. Thus a $\mathbb{Z}$-module is semisimple if and only if it is a direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, for various primes p, possibly repeating.

Next we claim that a $\mathbb{Z}$ module of the form $\mathbb{Z}/n\mathbb{Z}$ is semisimple if and only if n is square-free. Suppose $\mathbb{Z}/n\mathbb{Z}$ is a semisimple $\mathbb{Z}$-module. Let $n = p_1^{\alpha_1} \cdots p_i^{\alpha_i}$ be the unique factorization of n, with all p_j distinct. By the Chinese Remainder Theorem,

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$$

As noted above, $\mathbb{Z}/p_j^{\alpha_j}\mathbb{Z}$ is simple if and only if $\alpha_j = 1$. Furthermore, $\mathbb{Z}/p_j^{\alpha_j}\mathbb{Z}$ is semisimple if and only if $j = 1$, since if $j > 1$ then it contains $p_j\mathbb{Z}/p_j^{\alpha_j}\mathbb{Z}$ as a submodule with no orthogonal complement. So $\mathbb{Z}/n\mathbb{Z}$ is semisimple if and only if all the terms in the direct sum above are simple, which is to say, $\alpha_1, \ldots, \alpha_i = 1$, which is to say, if and only if n contains no repeated prime factors. This is equivalent to n being square-free.

Definition 3.32. A ring A is (left) **semisimple** if it is semisimple as a left module over itself.

Proposition 3.33. *Let A be a semisimple ring. Then*

1. *Every A-module is semisimple.*

2. *Every short exact sequence of A-modules splits.*

Proof. (1) Any free A-module is semisimple, since it is a direct sum of copies of A and A is semisimple. Any module is a quotient of a free module, and a quotient of a semisimple module is semisimple.

(2) Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence of A-modules. By (1), M is semisimple, so there is a submodule $L' \subset M$ such that $M = \operatorname{im} \alpha \oplus L \cong L \oplus L'$. Since $\operatorname{im} \alpha = \ker \beta$, $\beta|_{L'} : L' \to N$ is an isomorphism. Then $(\beta|_{L'})^{-1} : N \to M$ is a splitting for the sequence. □

Remark 3.34. If every short exact sequence of A-modules splits, then A is semisimple, so some sources give this as the definition of a simple ring.

Example 3.35. Any field or division ring is a semisimple ring.

Example 3.36. The ring $\mathbb{Z}$ is not semisimple. As we showed in example, $\mathbb{Z}$-module is semisimple if and only if it is a direct sum of prime order cyclic groups. But not every abelian group is such a direct sum, for example $\mathbb{Z}/4\mathbb{Z}$.

Theorem 3.37 (Artin-Wedderburn)**.** *Let A be a semisimple ring. Then*

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus \mathrm{M}_{n_r}(D_r)$$

for some integers $n_1, \ldots, n_r$ and division rings $D_1, \ldots, D_r$. Furthermore, this decomposition of A is unique up to permutatin the direct summands.

Proof. Involved. We aren't really going to use this, so we just include it for interest. □

Next we give some justification for the study of semisimple rings, showing that they arise naturally in the study of representations of finite groups.

Definition 3.38. Let K be a field. A **representation** of a group G is a K-vector space V along with a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Alternately, we can describe a representation in terms of the group algebra $K[G]$.

Definition 3.39. Let G be a finite group and K a field. The **group algebra** $K[G]$ is

$$K[G] := \left\{ \sum_{\sigma \in G} a_\sigma \sigma : a_\sigma \in K \right\}$$

As a K-vector space, it has a basis given by elements of G. The multiplication in $K[G]$ is determined by the multiplication in G along with K-linearity and the distributive property.

Remark 3.40. Given a representation $\rho : G \to \mathrm{GL}(V)$, for $\sigma \in G$ and $x \in V$ set $\sigma x := \rho(\sigma)x$. By linearity, extend this G-action on V to a $K[G]$-action. That is, extend ρ to

$$\widetilde{\rho} : K[G] \to \mathrm{End}(V) \qquad \left(\sum_{\sigma \in G} a_\sigma \sigma \right) x = \sum_{\sigma \in G} a_\sigma (\sigma x) = \sum_{\sigma \in G} a_\sigma \rho(\sigma) x$$

This gives V the structure of a $K[G]$-module. Conversely, given a $K[G]$-module V, we obtain an associated representation by reversing this procedure. So V being a representation is equivalent to being a $K[G]$-module.

Theorem 3.41 (Classical result in representation theory)**.** *Any representation of a finite group on a $\mathbb{C}$-vector space is completely reducible* [4]

[4] Complete reducible means a direct sum of irreducible representations. A representation is irreducible if it has no proper subrepresentations. A subrepresentation of a representation V is a subspace $W \subset V$ which is closed under the G-action.

In our terms, this theorem is just saying that $\mathbb{C}[G]$ is a semisimple ring. This generalizes as follows.

Theorem 3.42 (Mashke)**.** *Let G be a a finite group and K a field such that* $\operatorname{char} K$ *does not divide $|G|$. Then $K[G]$ is a semisimple ring.*

Before the proof, note that this immediately generalizes the classical result, since if $\operatorname{char} K = 0$ (such as the case $K = \mathbb{C}$), then it does not divide $|G|$ for any finite G.

Proof. Let $A = K[G]$. We show that any A-module is semisimple. Let M be an A-module, and $N \subset M$ a submodule. Considering M, N as K-vector spaces, there is a complementary subspace $N' \subset M$ such that

$$M = N \oplus N'$$

as K-vector spaces. Note that we have no information on whether N' is a submodule, so we are not done. Let $\pi : M \to N$ be the projection onto the first component. Note this is a map of K-vector spaces, but not necessarily of A-modules. Observe that

$$\pi|_N = \mathrm{Id}_N \qquad \pi^2 = \pi$$

To construct $N'' \subset M$ a submodule such that $M = N \oplus N''$, we "average" π over G. Define

$$\widetilde{\pi} : M \to N \qquad \widetilde{\pi}(m) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \left(\pi \left(gm \right) \right)$$

Since G is finite, the sum makes sense. Also note that $\frac{1}{|G|}$ is only allowed in the expression because we assumed that $\operatorname{char} K$ does not divide $|G|$. We claim that $\widetilde{\pi}$ satisfies

$$\widetilde{\pi}|_N = \mathrm{Id}_N \tag{3.1}$$
$$\widetilde{\pi}^2 = \widetilde{\pi} \tag{3.2}$$
$$\widetilde{\pi}(hm) = h\widetilde{\pi}(m) \qquad \forall m \in M, h \in G \tag{3.3}$$

The first equation is obvious because $\pi|_N = \mathrm{Id}_N$, and the second equation follows easily from the fact that $\pi^2 = \pi$. The last one requires a bit of a trick. Consider

$$\widetilde{\pi}(hm) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(ghm)$$

Make the substitution $\sigma = gh$. Then $\sigma^{-1} = h^{-1} g^{-1}$ and $g^{-1} = h\sigma^{-1}$. As g runs over G, so does σ, because right multiplication by h is an automorphism of G. So

$$\widetilde{\pi}(hm) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(ghm) = \frac{1}{|G|} \sum_{\sigma \in G} h\sigma^{-1} \pi(\sigma m)$$

But now we can pull out the h and switch back to g as our indexing variable, and it's clear that the expression on the right is $h\widetilde{\pi}(m)$.

$$\frac{1}{|G|} \sum_{\sigma \in G} h\sigma^{-1} \pi(\sigma m) = h \frac{1}{|G|} \sum_{\sigma \in G} \sigma^{-1} \pi(\sigma m) = h \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gm) = h\widetilde{\pi}(m)$$

So $\widetilde{\pi}$ is a map of A-modules. Then $N'' := \ker \widetilde{\pi}$ is an A-submodule of M. So consider the sequence

$$0 \to N'' \hookrightarrow M \xrightarrow{\widetilde{\pi}} N \to 0$$

This splits by the inclusion $N \hookrightarrow M$, so $M = N'' \oplus N$ as A-modules. Thus M is semisimple. □

Remark 3.43. Both the finiteness hypothesis and characteristic hypothesis were important steps in the proof above, so we can't drop them. In fact, one can prove that if G is finite and $\operatorname{char} K$ does divide $|G|$,that $K[G]$ is NOT semisimple. In another direction, there are infinite groups whose group ring is not semisimple, such as $G = \mathbb{Z}$. So Mashke's theorem is the best possible result in this direction.

3.3.2 Semisimple abelian categories

Lemma 3.44 (Splitting lemma)**.** *Let $\mathcal{A}$ be an abelian category and $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. The following are equivalent.*

1. *$B \to C$ has a right inverse.*
2. *$A \to B$ has a left inverse.*
3. *The short exact sequence is isomorphic (as a short exact sequence) to $0 \to A \to A \oplus C \to C \to 0$.*[5]

Proof. Just use Freyd-Mitchell embedding and the splitting lemma for R-modules. □

Definition 3.45. If the equivalent conditions above hold, we say the short exact sequence is **split**.

Definition 3.46. An abelian category $\mathcal{A}$ is **semisimple** if every short exact sequence in $\mathcal{A}$ is split.

Example 3.47. If R is a semisimple ring, the category of left R-modules is semisimple.

Definition 3.48. If X, Y are objects of $\mathcal{A}$, we say X is a **subobject** of Y if there is a monomorphism $X \to Y$. [6]

Definition 3.49. An object in an abelian category is **simple** if the only subobjects are the zero object and the object itself. An object is **semisimple** if it is a coproduct of simple objects.

[5] This is stronger than just $B \cong A \oplus C$ as objects. Isomorphism of short exact sequences means there is an isomorphism $B \to A \oplus C$ which fits into a suitable commutative diagram with the short exact sequences as rows.

[6] This is really not the proper definition of subobject. The right definition is way more complicated than it's worth though.

Remark 3.50. Some authors define a semisimple abelian category to be one in which every object is semisimple. This is not equivalent to our definition in terms of short exact sequences. If semisimple is used in this other sense, usually they would call a category in which every sequence splits a "split abelian category." It is possible to show that

$$\mathcal{A} \text{ is semisimple} \implies \mathcal{A} \text{ is split}$$

The converse is false, so our definition of semisimple category is weaker than the alternative definition. We will not work with the other definition, though, and we will continue to use "semisimple" to mean a category in which every short exact sequence splits.

Theorem 3.51. *If $\mathcal{A}$ is a semisimple abelian category, then it has a triangulated structure.*

Proof. Let $T = [1]$ be the identity functor on $\mathcal{A}$. We declare a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] = X$$

to be distinguished if it is exact at Y and Z and "exact at X" meaning $\ker f = \operatorname{im} h$.

Before verifying the triangulated axioms hold, we describe a form which every distinguished triangle in this category has. Given a distinguished triangle as above, we have a short exact sequence

$$0 \to \ker h \to Z \to \operatorname{im} h \to 0$$

Since $\mathcal{A}$ is semisimple, $Z \cong \ker h \oplus \operatorname{im} h$. Also $\ker h \cong \operatorname{im} g \cong \operatorname{coker} f$ and $\operatorname{im} h \cong \ker f$ by exactness, so $Z \cong \operatorname{coker} f \oplus f$. That is, we have an isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X \\ \downarrow{\scriptstyle \mathrm{Id}_X} & & \downarrow{\scriptstyle \mathrm{Id}_Y} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}_X} \\ X & \xrightarrow{f} & Y & \longrightarrow & \operatorname{coker} f \oplus \ker f & \longrightarrow & X \end{array}$$

All the morphisms in or out of $\operatorname{coker} f \oplus \ker f$ are induced by various universal properties of kernels, cokernels, and coproducts. So any triangle satisfying our definition of distinguished is isomorphic to a triangle of the form on the bottom row.

Now we can verify the triangulated axioms. (TR1a) and (TR1b) are obvious. For (TR1c), given $X \xrightarrow{f} Y$, it completes to the distinguished triangle $X \xrightarrow{f} Y \to \operatorname{coker} f \oplus \ker f \to X$. (TR2) is also obvious from the definition, since the definition is clearly symmetric with respect to X, Y, and Z. The first tricky verification is (TR3). Given a diagram as below with distinguished rows, we have to complete it to a morphism of triangles.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & \operatorname{coker} f \oplus \ker f & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & \operatorname{coker} f' \oplus \ker f' & \longrightarrow & X' \end{array}$$

Because of the commutativity of the first square, we have natural maps $\overline{h} : \operatorname{coker} f \to \operatorname{coker} f'$ and $\widetilde{g} : \ker f \to \ker f'$. Then the coproduct of these maps gives a map $\overline{h} \oplus \widetilde{g} : \operatorname{coker} f \oplus \ker f \to \operatorname{coker} f' \oplus \ker f'$. The resulting diagram then commutes. This is all very obvious if $\mathcal{A}$ is a category of R-modules, in the general case just be a more careful, or apply the Freyd-Mitchell embedding theorem.

Finally, we verify the octahedral axiom (TR4). Suppose we have a diagram below with commutative squares on the left and distinguished rows.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & \operatorname{coker} f \oplus \ker f & \longrightarrow & X \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{h} & Z & \longrightarrow & \operatorname{coker} h \oplus \ker h & \longrightarrow & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{g} & Z & \longrightarrow & \operatorname{coker} g \oplus \ker g & \longrightarrow & Y
\end{array}$$

We need vertical maps between the coker $\oplus$ ker terms to make a commutative diagram. As when verifying (TR3), we have naturally induced maps

$$\begin{aligned}
\overline{g} &: \operatorname{coker} f \to \operatorname{coker} h & \qquad i &: \ker f \to \ker h \\
\widetilde{f} &: \ker h \to \ker g & \qquad j &: \operatorname{coker} h \to \operatorname{coker} g
\end{aligned}$$

In the category of R-modules, $\overline{g}$ is the map induced on the quotients by g and $\widetilde{f}$ is just the restriction of f to $\ker h$, and i, j are inclusion maps. Then define

$$\begin{aligned}
u &: \operatorname{coker} f \oplus \ker f \to \operatorname{coker} h \oplus \ker h & \qquad u &= \overline{g} \oplus i \\
v &: \operatorname{coker} h \oplus \ker h \to \operatorname{coker} g \oplus \ker g & \qquad v &= j \oplus \widetilde{f}
\end{aligned}$$

Then we claim that the following diagram commutes and that the bottom row is distinguished.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{a} & \operatorname{coker} f \oplus \ker f & \longrightarrow & X \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle u} & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{h} & Z & \xrightarrow{b} & \operatorname{coker} h \oplus \ker h & \longrightarrow & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \vdots{\scriptstyle v} & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{g} & Z & \longrightarrow & \operatorname{coker} g \oplus \ker g & \longrightarrow & Y \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle a} \\
\operatorname{coker} f \oplus \ker f & \overset{u}{\cdots\rightarrow} & \operatorname{coker} h \oplus \ker h & \overset{v}{\cdots\rightarrow} & \operatorname{coker} g \oplus \ker g & \xrightarrow{0} & \operatorname{coker} f \oplus \ker f
\end{array}$$

Commutativity is not hard to check, just carry things out in a suitable category of R-modules using the Freyd-Mitchell embedding theorem. We work through verifying exactnes of the resulting sequence at the $\operatorname{coker} h \oplus \ker h$ term, again working as if everything was an R-module.

Since $v = j \oplus \widetilde{f}$ and j is a monomorphism, $\ker v = \ker i \oplus \ker \widetilde{f} = 0 \oplus \ker f$. Similarly, since $u = \overline{g} \oplus i$, $\operatorname{im} u = \operatorname{im} \overline{g} \oplus \ker f$. So it suffices to verify that $\operatorname{im} \overline{g} \cap \operatorname{coker} h = 0$. Since $gf = h$, $\operatorname{im} g \subset \operatorname{im} h$, so $\overline{g}$ maps to zero into $\operatorname{coker} h$. Thus $\operatorname{im} \overline{g} \oplus \operatorname{coker} h = 0$ so $\operatorname{im} u = 0 \oplus ker f = \ker v$. So the constructed sequence is exact at that term. We leave it to you to check exactness at the other terms, since they are easier. $\square$

3.4 Abelian triangulated category is semisimple

Our next object is to show that any abelian triangulated category is semisimple. This result is not interesting because abelian triangulated categories come up often - in fact, the very opposite. The result says that abelian categories and triangulated categories are structures which are not particularly compatible.

While it is possible for a category to be both, having both forces a very rigid structure, that of being semisimple. Being semisimple is such a stringent "condition that this essentially says that in any "typical" situation, a category cannot be both abelian and triangulated. Before the main result, we need some more general facts about triangulated categories.

Proposition 3.52 (Direct sum of distinguished triangles is distinguished)**.** *Let $\mathcal{C}$ be a triangulated category and suppose*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

are distinguished triangles. The triangle

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{v \oplus v'} Z \oplus Z' \xrightarrow{w \oplus w'} X[1] \oplus X'[1]$$

is distinguished. Note that since the translation functor $[1]$ *is additive, there is a natural isomorphism* $(X \oplus X')[1] \cong X[1] \oplus X'[1]$.

Proof. By (TR1c), we can complete $X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y'$ to a distinguished triangle.

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \to U \to X[1] \oplus X'[1]$$

Let $\pi_X : X \oplus X' \to X$ and $\pi_Y : Y \oplus Y' \to Y$ be the canonical "projection" maps associated with the coproduct. Then we have a commutative diagram

$$\begin{array}{ccccccc}
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \longrightarrow & U & \longrightarrow & X[1] \oplus X'[1] \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} & & & & \downarrow{\scriptstyle \pi_X[1]} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]
\end{array}$$

Since the rows are distinguished, by (TR3) there is a map $f : U \to Z$ completing this to a morphism of triangles.

$$\begin{array}{ccccccc}
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \longrightarrow & U & \longrightarrow & X[1] \oplus X'[1] \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi_X[1]} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]
\end{array}$$

Similarly, considering the projections $\pi_{X'} : X \oplus X' \to X'$ and $\pi_{Y'} : Y \oplus Y' \to Y'$ and again using (TR3), there is a map $f' : U \to Z'$ fitting into the diagram below.

$$\begin{array}{ccccccc}
X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \longrightarrow & U & \longrightarrow & X[1] \oplus X'[1] \\
\downarrow{\scriptstyle \pi_{X'}} & & \downarrow{\scriptstyle \pi_{Y'}} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \pi_{X'}[1]} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]
\end{array}$$

Set $\phi = f \oplus f' : U \to Z \oplus Z'$. Note that $\mathrm{Id}_{X \oplus X'} = \pi_X \oplus \pi_{X'}$, so ϕ fits into the commutative diagram

$$\begin{array}{ccccccc} X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \longrightarrow & U & \longrightarrow & X[1] \oplus X'[1] \\ \downarrow 1 & & \downarrow 1 & & \vdots \phi & & \downarrow 1 \\ X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{v \oplus v'} & Z \oplus Z' & \xrightarrow{w \oplus w'} & X[1] \oplus X'[1] \end{array}$$

It is tempting to apply lemma 3.18 here, but that only applies if we know both rows are distinguished. However, a slight tweak of the argument for lemma 3.18 shows that ϕ is an isomorphism, using the fact that the Hom functor commutes with finite direct sums. We omit the details. Since ϕ is an isomorphism, this diagram above is an isomorphism of triangles, so by (TR1b) the bottom rows is also distinguished. □

Corollary 3.53. *Let $\mathcal{C}$ be a triangulated category with objects X, Y. Let $i : X \to X \oplus Y$ and $p : X \oplus Y \to Y$ be the canonical maps associated with the biproduct. Then*

$$X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1]$$

is a distinguished triangle.

Proof. By (TR1a), $X \xrightarrow{1} X \to 0 \to X[1]$ and $Y \xrightarrow{1} Y \to 0 \to Y[1]$ are distinguished. Apply the rotation axiom (TR2) to the second to get that $0 \to Y \xrightarrow{1} Y \to 0$ is distinguished. Then apply proposition 3.52 to get the desired distinguished triangle. □

Corollary 3.54. *Suppose $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{0} X[1]$ is distinguished. Then it is isomorphic as a triangle to a triangle of the form $X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1]$.*

Proof. Rotate both triangles using (TR2) and then consider the diagram

$$\begin{array}{ccccccc} X[1] & \xrightarrow{0} & X & \xrightarrow{u} & Z & \xrightarrow{v} & Y \\ \downarrow 1 & & \downarrow 1 & & & & \downarrow 1 \\ X[1] & \xrightarrow{0} & X & \xrightarrow{i} & X \oplus Y & \xrightarrow{p} & Y \end{array}$$

Then by (TR3) this completes to a morphism of triangles. By the 5-lemma 3.18, the new arrow is an isomorphism, so this is an isomorphism of triangles. Then rotate back to get the isomorphism between the original triangles. □

Remark 3.55. As a consequence of the previous results, we can describe all monomorphisms and epimorphisms in a triangulated category. Morally speaking, all monomorphisms are all essentially inclusions $X \to X \oplus Y$ and all epimorphisms are projections $X \oplus Y \to Y$, up to an automorphism. The next result makes this statement precise.

Proposition 3.56. *Let $\mathcal{C}$ be a triangulated category.*

1. *If $X \xrightarrow{f} Y$ is a monomorphism in $\mathcal{C}$, then there exists an isomorphism $Z \xrightarrow{\phi} X \oplus Y$ such that $f = \phi i$, where $X \xrightarrow{i} X \oplus Y$ is the canonical map. In particular, $p\phi^{-1}$ is a left inverse for f, where $X \oplus Z \xrightarrow{p} X$ is the canonical map.*

2. *If $X \xrightarrow{f} Y$ is an epimorphism in $\mathcal{C}$, then there exists an isomorphism $X \xrightarrow{\psi} Y \oplus Z$ such that $f = p\psi$, where $Y \oplus Z \xrightarrow{p} Y$ is the canonical map. In particular, $\psi^{-1}i$ is a right inverse for f, where $Y \xrightarrow{i} Y \oplus Z$ is the canonical map.*

Proof. (1) By (TR1c) we can extend f to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. Then apply (TR2) to rotate it to a distinguished triangle $Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$. We know $f \circ h[-1] = 0$, so since f is a monomorphism $h[-1] = 0$, which implies $h = 0$. Then by the previous corollary there is an isomorphism $Z \xrightarrow{\phi} X \oplus Y$ making an isomorphism of triangles. In particular, the resulting commutative diagram implies $f = \phi i$.

(2) Reverse some arrows in the argument for (1). □

Remark 3.57. As a slogan, we memorialize the previous result by saying

All monomorphisms and epimorphisms in a triangulated category are split.

Theorem 3.58. *An abelian triangulated category is semisimple.*

Proof. Let $\mathcal{C}$ be an abelian triangulated category, and let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{C}$. Then f is a monomorphism, so by the previous result f splits. That is, f has a left inverse, which means the sequence splits. That is, $Y \cong X \oplus Z$ fitting into an isomorphism of exact sequences. Thus $\mathcal{C}$ is semisimple. □

3.5 Homotopy category is triangulated

Our next goal is to show that the homotopy category $K(\mathcal{A})$ of an additive category $\mathcal{A}$ is triangulated. Recall that if $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism of complexes, then the cone of f is the complex $C_f^\bullet$ where

$$C_f^n = X^n[1] \oplus Y^n = X^{n+1} \oplus Y^n \qquad d_{C_f}^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}$$

and that there are always morphisms of complexes $i_f^\bullet : Y^\bullet \to C_f^\bullet$ and $p_f^\bullet : C_f^\bullet \to X^\bullet[1]$, which are just canonical maps associated with the biproduct on each term.

Definition 3.59. A **standard triangle** in $C(\mathcal{A})$ is a triangle of the form

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i_f^\bullet} C_f^\bullet \xrightarrow{p_f^\bullet} X^\bullet[1]$$

The rough idea of defining distinguished triangles in $K(\mathcal{A})$ is to set these standard triangles to be distinguished. Note that immediately we can see that this will not work to put a triangulated structure on the chain complex category $C(\mathcal{A})$, because it fails (TR1a). The cone of the identity morphism $X^\bullet \to X^\bullet$ is not zero, so the triangle

$$X^\bullet \xrightarrow{\mathrm{Id}_{X^\bullet}} X^\bullet \to 0 \to X^\bullet[1]$$

is not a standard triangle, nor is it isomorphic to a standard triangle using morphisms of $C(\mathcal{A})$. However, as we'll see soon, the greater flexibility afforded by morphisms up to homotopy allows such a triangle to be isomorphic to a standard triangle.

Definition 3.60. A triangle in $K(\mathcal{A})$ is **distinguished** if it is isomorphic (in $K(\mathcal{A})$) to a standard triangle.[7]

Lemma 3.61 (TR1a)**.** *Let $X^\bullet$ be a complex in $C(\mathcal{A})$. Then the cone $C_{\mathrm{Id}_{X^\bullet}}$ is isomorphic to the zero object in $K(\mathcal{A})$.*

Proof. First, note that to prove an object is isomorphic to the zero object in an additive category, it suffices to show that the identity morphism of that object is zero. So we will just show that the identity morphism of $C = C_{\mathrm{Id}_{X^\bullet}}$ is nullhomotopic. First recall that the boundary maps for C are given by

$$d_C^n : X^{n+1} \oplus X^n \to X^{n+2} \oplus X^{n+1} \qquad \begin{pmatrix} -d_X^{n+1} & 0 \\ \mathrm{Id}_{X^{n+1}} & d_X^n \end{pmatrix}$$

Define

$$h^n : X^{n+1} \oplus X^n \to X^n \oplus X^{n-1} \qquad \begin{pmatrix} 0 & \mathrm{Id}_{X^n} \\ 0 & 0 \end{pmatrix}$$

In other words, $h(x, y) = (y, 0)$. To verify that Id_C is nullhomotopic, we need to show that $d_C^{n-1} h^n + h^{n+1} d_C^n = \mathrm{Id}_{C^n}$. This is a cozy matrix calculation.

$$\begin{aligned} d_C^{n-1} h^n + h^{n+1} d_n^C &= \begin{pmatrix} -d_X^n & 0 \\ \mathrm{Id}_{X^n} & d_X^{n-1} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id}_X^n \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{Id}_{X^{n+1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ \mathrm{Id}_{X^{n+1}} & d_X^n \end{pmatrix} \\ &= \begin{pmatrix} 0 & -d_X^n \\ 0 & \mathrm{Id}_{X^n} \end{pmatrix} + \begin{pmatrix} \mathrm{Id}_{X^{n+1}} & d_X^n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{X^{n+1}} & 0 \\ 0 & \mathrm{Id}_{X^n} \end{pmatrix} = \mathrm{Id}_{C^n} \end{aligned}$$

□

Lemma 3.62 (TR3)**.** *Suppose we have a diagram in $C(\mathcal{A})$ as below.*

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet \\ \downarrow{\scriptstyle u^\bullet} & & \downarrow{\scriptstyle v^\bullet} \\ X_1^\bullet & \xrightarrow{g^\bullet} & Y_1^\bullet \end{array}$$

Then there exists a morphism $w^\bullet : C_f \to C_g$, fitting into the following diagram.

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{i_f^\bullet} & C_f & \xrightarrow{p_f^\bullet} & X^\bullet[1] \\ \downarrow{\scriptstyle u^\bullet} & & \downarrow{\scriptstyle v^\bullet} & & \downarrow{\scriptstyle w^\bullet} & & \downarrow{\scriptstyle u^\bullet[1]} \\ X_1^\bullet & \xrightarrow{g^\bullet} & Y_1^\bullet & \xrightarrow{i_g^\bullet} & C_g & \xrightarrow{p_g^\bullet} & X_1^\bullet[1] \end{array}$$

If the first diagram commutes, then the second also commutes. If the first diagram commutes up to homotopy, then the second also commutes up to homotopy.

[7] Technically, instead of "standard triangle" we should say the image of a standard triangle in $K(\mathcal{A})$, but whatever.

Proof. Suppose the first diagram commutes up to homotopy, so there are maps

$$h^n : X^n \to Y_1^{n+1}$$

such that

$$g^n u^n - v^n f^n = d_{Y_1}^{n-1} h^n + h^{n+1} d_X^n$$

If the first diagram commutes, then the equation is true with $h = 0$. Then define

$$w^\bullet : C_f C_g \qquad w^n : X^{n+1} \oplus Y^n \to X_1^{n+1} \oplus Y_1^n \qquad w^n = \begin{pmatrix} u^{n+1} & 0 \\ -h^{n+1} & v^n \end{pmatrix}$$

When $h = 0$ it is a straightforward calculation to see that the squares in the second diagram commute. When $h \neq 0$, showing the remaining squares commute up to homotopy is also not very interesting, so we omit the calculation. □

Lemma 3.63. *Suppose (TR1), (TR2), and (TR3) hold in $K(\mathcal{A})$. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $K(\mathcal{A})$ and let $a^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$ which represents $f^\bullet$. The following are equivalent.*

1. *The triangle $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \to Z^\bullet \to X^\bullet[1]$ is distinguished.*
2. *There exists an isomorphism $u^\bullet : Z^\bullet \to C_a$ in $K(\mathcal{A})$ making the following diagram commute (in $K(\mathcal{A})$).*

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u^\bullet} & & \downarrow{\scriptstyle 1} \\
X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \xrightarrow{i_a^\bullet} & C_a & \xrightarrow{p_a^\bullet} & X^\bullet[1]
\end{array}$$

Proof. (2) $\implies$ (1) is immediate from the definition of distinguished triangles in $K(\mathcal{A})$, so we just need to prove (1) $\implies$ (2). By definition, the triangle involving the cone C_a is distinguished. By (TR3), the diagram below can be completed to a morphism of triangles.

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} \\
X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \xrightarrow{i_a^\bullet} & C_a & \xrightarrow{p_a^\bullet} & X^\bullet[1]
\end{array}$$

So a morphism $u^\bullet$ exists. By remark 3.19, the triangulated 5-lemma holds in $K(\mathcal{A})$, so $u^\bullet$ must be an isomorphism. □

Theorem 3.64. *Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ with the translation and distinguished triangles above is a triangulated category.*

Proof. (TR1b) and (TR1c) are obvious from the definition. To verify (TR1a), we claim that the following is an isomorphism of triangles

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{\mathrm{Id}} & X^\bullet & \longrightarrow & 0 & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\
X^\bullet & \xrightarrow{\mathrm{Id}} & X^\bullet & \xrightarrow{i^\bullet_{\mathrm{Id}}} & C_{\mathrm{Id}} & \xrightarrow{p^\bullet_{\mathrm{Id}}} & X^\bullet[1]
\end{array}$$

By lemma 3.61, the map $0 \to C_{\mathrm{Id}}$ is an isomorphism. The only thing to verify here is that the middle square commutes in the homotopy category. That is, we need to verify that $i^\bullet_{\mathrm{Id}} : X^\bullet \to C_{\mathrm{Id}}$ is nullhomotopic. The nullhomotopy is given by maps

$$h^n : X^n \to C^{n-1} = X^n \oplus X^{n-1} \qquad x \mapsto (x, 0)$$

and we omit the calculation to show that this gives a nullhomotopy of $i^\bullet_{\mathrm{Id}}$. This finishes the verification of (TR1a). (TR3) is immediate lemma 3.62.

Next we verify (TR2). Suppose we have a standard triangle.

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{i^\bullet_f} C_f \xrightarrow{p^\bullet_f} X^\bullet[1]$$

To prove (TR2), we need to prove that the rotated version is distinguished (we also need to show that it's possible to rotate the other way, but we'll discuss that later). The rotated triangle is

$$Y^\bullet \xrightarrow{i^\bullet_f} C_f \xrightarrow{p^\bullet_f} X^\bullet[1] \xrightarrow{-f^\bullet[1]} Y^\bullet[1]$$

We will show this is isomorphic to the standard triangle associated with $i^\bullet_f$. That is, we have the standard triangle below where C_{i_f} is the cone of i_f.

$$Y^\bullet \xrightarrow{i^\bullet_f} C_f \xrightarrow{i^\bullet_{i_f}} C_{i_f} \xrightarrow{p^\bullet_{i_f}} Y^\bullet[1]$$

For concreteness, C_{i_f} is the complex with objects

$$C^n_{i_f} = Y^\bullet[1]^n \oplus C^n_f = Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

and differentials

$$d^n_{C_{i_f}} = \begin{pmatrix} -d^{n+1}_Y & 0 \\ i^{n+1}_f & d^n_{C_f} \end{pmatrix} = \begin{pmatrix} -d^{n+1}_Y & 0 & 0 \\ 0 & -d^{n+1}_X & 0 \\ \mathrm{Id}^{n+1}_Y & f^{n+1} & d^n_Y \end{pmatrix}$$

and we have the maps

$$\begin{aligned} i^n_{i_f} &: C^n_f \to C^n_{i_f} \qquad (x_{n+1}, y_n) \mapsto (0, x_{n+1}, y_n) \\ p^n_{i_f} &: C^n_{i_f} \to Y^{n+1} \qquad (y_{n+1}, x_{n+1}, y_n) \mapsto y_{n+1} \end{aligned}$$

As we said before, we claim that the rotated triangle is isomorphic to the standard triangle involving the cone C_{i_f}. To this end, define

$$\alpha^n : X^{n+1} \to C^n_{i_f} \qquad x \mapsto (-f(x), x, 0) \qquad \begin{pmatrix} -f^{n+1} \\ \mathrm{Id}_{X^{n+1}} \\ 0 \end{pmatrix}$$

$$\beta^n : C^n_{i_f} \to X^{n+1} \qquad (y_{n+1}, x_{n+1}, y_n) \mapsto x_{n+1} \qquad \begin{pmatrix} 0 & \mathrm{Id}_{X^{n+1}} & 0 \end{pmatrix}$$

One can check that $\alpha^\bullet, \beta^\bullet$ are morphisms of complexes $X^\bullet[1] \to C^\bullet_{i_f}$ and $C^\bullet_{i_f} \to X^\bullet[1]$ respectively. We claim they are inverses up to homotopy, hence inverses in $K(\mathcal{A})$ between $X^\bullet[1]$ and $C^\bullet_{i_f}$ in $K(\mathcal{A})$. In one direction, the composition is equal to the identity even in $C(\mathcal{A})$.

$$\beta^n\alpha^n = \begin{pmatrix} 0 & \mathrm{Id}_{X^{n+1}} & 0 \end{pmatrix} \begin{pmatrix} -f^{n+1} \\ \mathrm{Id}_{X^{n+1}} \\ 0 \end{pmatrix} = \mathrm{Id}_{X^{n+1}}$$

However, the reverse composition is not equal to the identity as a chain map, but only up to homotopy, as we now show. Define

$$h^n : C^n_{i_f} \to C^n_{i_f} \qquad (y_{n+1}, x_{n+1}, y_n) \mapsto (y_n, 0, 0) \qquad \begin{pmatrix} 0 & 0 & \mathrm{Id}_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then one can do some matrix multiplications to verify that

$$\mathrm{Id}_{C^n_{i_f}} - \alpha^n\beta^n = d^n_{C_{i_f}} h^n + h^{n+1} d^n_{C_{i_f}}$$

hence h is a homotopy, so $\alpha^n\beta^n$ is homotopic to the identity. So $\alpha^\bullet\beta^\bullet$ and $\beta^\bullet\alpha^\bullet$ are both the respective identities in $K(\mathcal{A})$, so $\alpha^\bullet$ is an isomorphism $X^\bullet[1] \to C^\bullet_{i_f}$. All that remains to verify is that it fits into a diagram making an isomorphism of triangles as below.

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{i^\bullet_f} & C_f & \xrightarrow{p^\bullet_f} & X^\bullet[1] & \xrightarrow{-f^\bullet[1]} & Y^\bullet[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha^\bullet} & & \downarrow{\scriptstyle 1} \\ Y^\bullet & \xrightarrow{i^\bullet_f} & C_f & \xrightarrow{i^\bullet_{i_f}} & C_{i_f} & \xrightarrow{p^\bullet_{i_f}} & Y^\bullet[1] \end{array}$$

The left square obviously commutes, and the right square is also commutes even in the complex category. The middle square however only commutes up to homotopy, as we now verify. Consider the composition

$$\beta^n i^n_{i_f} = \begin{pmatrix} 0 & \mathrm{Id}_{X^{n+1}} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{Id}_{X^{n+1}} & 0 \\ 0 & 0 & \mathrm{Id}_{Y^n} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id}_{X^{n+1}} & 0 \end{pmatrix} = p^n_f$$

Thus $\alpha^\bullet\beta^\bullet i^\bullet_{i_f} = \alpha^\bullet p^\bullet_f$. But $\alpha^\bullet\beta^\bullet$ is homotopic to $\mathrm{Id}_{X^\bullet[1]}$, so in the homotopy category $\alpha^\bullet p^\bullet_f = i^\bullet_{i_f}$, which is precisely the commutativity of the middle square we need.

This completes the proof that if we rotate a standard distinguished triangle in one direction, it remains distinguished, but this is not the entirety of (TR2). (TR2) also requires that if the rotated triangle is distinguished, then the original triangle was distinguished, or equivalently, if we start with a standard distinguished triangle, then rotating it the other direction also gives a distinguished triangle.

We omit the argument for this rotation direction, since it is analogous to the previous argument. The argument follows the same basic steps: write down both the rotated triangle and the associated standard triangle associated to the first map in it, which is $-p^\bullet_f[-1]$. Then show these triangles are isomorphic, by constructing chain maps in both directions such that the compositions are homotopic to the identity.

Finally, we verify the octahedral axiom (TR4). We start with the following diagram with distinguished rows and commutative squares on the left.

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g^\bullet} & & & & \downarrow{\scriptstyle 1} \\
X^\bullet & \xrightarrow{h^\bullet} & Z^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle f^\bullet[1]} \\
Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \longrightarrow & X_1^\bullet & \longrightarrow & Y^\bullet[1]
\end{array}$$

By lemma 3.63, the distinguished rows are isomorphic to triangles of cones for representative chain maps. In particular, we apply the lemma to the top and bottom rows to obtain isomorphisms of triangles

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z_1^\bullet & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle 1} \\
X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \xrightarrow{i_a^\bullet} & C_a & \xrightarrow{p_a^\bullet} & X^\bullet[1]
\end{array}
\qquad
\begin{array}{ccccccc}
Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \longrightarrow & Y_1^\bullet & \longrightarrow & Y^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle 1} \\
Y^\bullet & \xrightarrow{b^\bullet} & Z^\bullet & \xrightarrow{i_b^\bullet} & C_b & \xrightarrow{p_b^\bullet} & Y^\bullet[1]
\end{array}$$

Define $c^\bullet : X^\bullet \to Z^\bullet$ by $c^\bullet = b^\bullet a^\bullet$. Since $a^\bullet$ represents $f^\bullet$ and $b^\bullet$ represents $g^\bullet$, $c^\bullet$ represents $h^\bullet$. So again using lemma 3.63, the middle triangle is isomorphic to the cone triangle for $c^\bullet$. So we have the following diagram in $C(\mathcal{A})$, where the left squares commute (not up to homotopy, literally commute).

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \longrightarrow & C_a & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b^\bullet} & & & & \downarrow{\scriptstyle 1} \\
X^\bullet & \xrightarrow{c^\bullet} & Z^\bullet & \longrightarrow & C_c & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle a^\bullet} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle f^\bullet[1]} \\
Y^\bullet & \xrightarrow{b^\bullet} & Z^\bullet & \longrightarrow & C_b & \longrightarrow & Y^\bullet[1]
\end{array}$$

Since the left squares commute, we can define $u^\bullet : C_a \to C_c$ and $v^\bullet : C_c \to C_b$ making the whole diagram commute.

$$u^n : C_a^n \to C_c^n \qquad u^n = \begin{pmatrix} \mathrm{Id}_{X^{n+1}} & 0 \\ 0 & b^n \end{pmatrix}$$

$$v^n : C_c^n \to C_b^n \qquad v^n = \begin{pmatrix} a^{n+1} & 0 \\ 0 & \mathrm{Id}_{Z^n} \end{pmatrix}$$

So we now have the following commutative diagram in $C(\mathcal{A})$.

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \longrightarrow & C_a & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b^\bullet} & & \downarrow{\scriptstyle u^\bullet} & & \downarrow{\scriptstyle 1} \\
X^\bullet & \xrightarrow{c^\bullet} & Z^\bullet & \longrightarrow & C_c & \longrightarrow & X^\bullet[1] \\
\downarrow{\scriptstyle a^\bullet} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle v^\bullet} & & \downarrow{\scriptstyle f^\bullet[1]} \\
Y^\bullet & \xrightarrow{b^\bullet} & Z^\bullet & \longrightarrow & C_b & \longrightarrow & Y^\bullet[1]
\end{array}$$

We just need to verify that $C_a \xrightarrow{u^\bullet} C_c \xrightarrow{v^\bullet} C_a[1]$ is a distinguished triangle, where the map $C_b \to C_a[1]$ is $i_a^\bullet[1] \circ p_b^\bullet$. We will show that it is isomorphic to the cone triangle for $u^\bullet$. So consider the diagram

$$\begin{array}{ccccccc}
C_a & \xrightarrow{u^\bullet} & C_c & \xrightarrow{v^\bullet} & C_b & \xrightarrow{i_a^\bullet[1]\circ p_b^\bullet} & C_a[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} \\
C_a & \xrightarrow{u^\bullet} & C_c & \xrightarrow{i_u^\bullet} & C_u & \xrightarrow{p_u^\bullet} & C_a[1]
\end{array}$$

Define $w^\bullet : C_b \to C_u$ by

$$w^n : C_b^n \to C_u^n \qquad w^n : Y^{n+1} \oplus Z^n \to X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus Z^n$$

$$w^n = \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{Id}_{Z^n} \end{pmatrix} \qquad w^n(y_{n+1}, z_n) = (0, y_{n+1}, 0, z_n)$$

We omit the details, but it is similar to other arguments we have done to show that $w^\bullet$ fits into the preceding diagram to make a morphism of triangles, and is an isomorphism in $K(\mathcal{A})$. Hence the triangle $C_a \to C_c \to C_b$ is distinguished, completing the proof of (TR4). ☐

We now have the tools to give an example where the homotopy category fails to be abelian.

Corollary 3.65. *Let* $\mathcal{A} = \mathrm{AbGp}$ *be the category of abelian groups. The category* $K(\mathcal{A})$ *is not abelian.*

Proof. Let $p \in \mathbb{Z}$ be a prime, and consider the nonsplit short exact sequence of abelian groups

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{1 \mapsto p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \to 0 \tag{3.4}$$

where π is the quotient map $1 \mapsto 1 \bmod p$. Recall the additive functors $C : \mathcal{A} \to C(\mathcal{A})$ which takes an object to the complex with that object concentrated in degree zero, and $H : C(\mathcal{A}) \to K(\mathcal{A})$ which does nothing to objects and takes a morphism to its homotopy class. Let $K = H \circ C : \mathcal{A} \to K(\mathcal{A})$ be the composition, and apply K to π.

$$K(\mathbb{Z}/p^2\mathbb{Z}) \xrightarrow{K\pi} K(\mathbb{Z}/p\mathbb{Z})$$

It is not hard to show that $K\pi$ is an epimorphism, and that $K\pi \neq 0$, which is to say, $C\pi$ is not nullhomotopic. Now suppose to the contrary that $K(\mathcal{A})$ is abelian. Then we have a short exact sequence in $K(\mathcal{A})$

$$0 \to \ker(K\pi) \to K(\mathbb{Z}/p^2\mathbb{Z}) \xrightarrow{K\pi} K(\mathbb{Z}/p\mathbb{Z}) \to 0$$

By theorem 3.64 $K(\mathcal{A})$ is triangulated, so by theorem 3.58, $K(\mathcal{A})$ is semisimple. Hence this sequence splits, which is to say, there is a morphism $s : K(\mathbb{Z}/p\mathbb{Z}) \to K(\mathbb{Z}/p^2\mathbb{Z})$ in $K(\mathcal{A})$ such that $K\pi \circ s = \mathrm{Id}_{K(\mathbb{Z}/p\mathbb{Z})}$. That is, there is a morphism of complexes $\widetilde{s} : C(\mathbb{Z}/p\mathbb{Z}) \to C(\mathbb{Z}/p^2\mathbb{Z})$ where $H\widetilde{s} = s$ and such that $C\pi \circ \widetilde{s}$ is homotopic to the identity on $C(\mathbb{Z}/p\mathbb{Z})$.

However, as $C(\mathbb{Z}/p\mathbb{Z})$ and $C(\mathbb{Z}/p^2\mathbb{Z})$ are both concentrated in degree zero, there are no nontrivial homotopies. Thus $C\pi \circ \widetilde{s} = \mathrm{Id}_{C(\mathbb{Z}/p\mathbb{Z})}$. In degree zero, we have $(C\pi)^0 = \pi$ and so $\pi \circ (\widetilde{s})^0 = \mathrm{Id}_{\mathbb{Z}/p\mathbb{Z}}$. That is, $(\widetilde{s})^0$ gives a splitting of sequence 3.4, which is a contradiction since that sequence is not split. □

Remark 3.66. The argument above is easily adapted to the more general situation where $\mathcal{A}$ is any abelian category which is not semisimple. That is, the homotopy category over any non-semisimple abelian category is not abelian.

Above we showed that $K(\mathcal{A})$ is triangulated, which depended only upon $\mathcal{A}$ being an additive category. Now suppose that $\mathcal{A}$ is also abelian, so that we have the cohomology functors

$$H^n : C(\mathcal{A}) \to \mathcal{A} \qquad X^\bullet \mapsto H^n X^\bullet$$

Since homotopy equivalent chain maps induce the same maps on H^n, this induces a functor on $K(\mathcal{A})$.

$$H^n : K(\mathcal{A}) \to \mathcal{A} \qquad X^\bullet \mapsto H^n X^\bullet$$

Theorem 3.67. *Let $\mathcal{A}$ be an abelian category. The functor $H^n : K(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor.*

Proof. It's enough to show that H^0 is cohomological, and then use translation. We need to show that for any distinguished triangle $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} X^\bullet[1]$ in $K(\mathcal{A})$, the sequence

$$H^0 X^\bullet \xrightarrow{H^0 f^\bullet} H^0 Y^\bullet \xrightarrow{H^0 g^\bullet} H^0 Z^\bullet$$

is an exact sequence in $\mathcal{A}$. By lemma 3.63, our triangle is isomorphic to a cone triangle for $a^\bullet$ where $a^\bullet$ is a representative of $d^\bullet$.

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\ \downarrow 1 & & \downarrow 1 & & \cong \downarrow u^\bullet & & \downarrow 1 \\ X^\bullet & \xrightarrow{a^\bullet} & Y^\bullet & \xrightarrow{i_a^\bullet} & C_a & \xrightarrow{p_a^\bullet} & X^\bullet[1] \end{array}$$

Now apply the functor H^0 to this to get a commutative diagram in $\mathcal{A}$.[8]

$$\begin{array}{ccccccc} H^0 X^\bullet & \xrightarrow{H^0 f^\bullet} & H^0 Y^\bullet & \longrightarrow & H^0 Z^\bullet & \longrightarrow & H^0 X^\bullet[1] \\ \downarrow 1 & & \downarrow 1 & & \cong \downarrow H^0 u^\bullet & & \downarrow 1 \\ H^0 X^\bullet & \xrightarrow{H^0 a^\bullet} & H^0 Y^\bullet & \xrightarrow{H^0 i_a^\bullet} & H^0 C_a & \xrightarrow{H^0 p_a^\bullet} & H^0 X^\bullet[1] \end{array}$$

[8]In case you're worried about whether $H^0 X^\bullet[1]$ is $H^0(X^\bullet[1])$ or $(H^0 X^\bullet)[1]$, remember that H^0 commutes with translation, meaning that these two are naturally isomorphic.

We wanted exactness of the top, but because the vertical arrows are all isomorphisms, this is equivalent to exactness of the bottom row. Recall we have a short exact sequence in $C(\mathcal{A})$

$$0 \to Y^\bullet \xrightarrow{i_a^\bullet} C_a \xrightarrow{p_a^\bullet} X^\bullet[1] \to 0$$

which then induces a long exact sequence in cohomology, which includes the segment

$$H^{-1}X^\bullet[1] \xrightarrow{\partial} H^0Y^\bullet \xrightarrow{H^0 i_a^\bullet} H^0C_a$$

But we know that $H^{-1}X^\bullet[1] \cong H^0X^\bullet$, and furthermore by tracing through the construction of ∂ in the snake lemma, one can show that the connecting homomorphism ∂ here is the same as $H^0a^\bullet$. So the sequence

$$H^0 \xrightarrow{H^0 a^\bullet} H^0Y^\bullet \xrightarrow{H^0 i_a^\bullet} H^0C_a$$

is exact (in $\mathcal{A}$), so the sequence $H^0X^\bullet \xrightarrow{H^0 f^\bullet} H^0Y^\bullet \to H^0Z^\bullet$ is also exact. Thus H^0 is cohomological. $\square$

4 Localization

Our next goal is to construct the derived category from the homotopy category using a process of localizing a category. This is akin to localizing a ring, so we motivate this by first studying some localization of rings, even noncommutative rings.

4.1 Localization of rings

4.1.1 Commutative rings

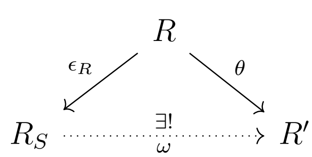
First we discuss the commutative case. Let R be a commutative (unital, associative) ring. Let $S \subset R \setminus \{0\}$ be a multiplicative subset, meaning $1 \in S$ and S is closed under multiplication. Then there exists a commutative (unital, associative) ring R_S and a ring homomorphism

$$\epsilon_S : R \to R_S$$

such that

1. $\epsilon(s) \in R_S^\times$ for all $s \in S$, and
2. ϵ_S is universal with this property.

Property 2 above means that for any ring homomorphism $\theta : R \to R'$ such that $\theta(s) \in (R')^\times$ for all $s \in S$, there exists a unique ring homomorphism $\omega : R_S \to R'$ such that the following diagram commutes.



In the usual way, this universal property makes R_S unique up to isomorphism. The ring R_S is called the **localization of *R* at *S***. This ring has two other key properties which are very useful when working with localizations.

(A) Every element of R_S is of the form $\epsilon_S(r)\epsilon_S(s)^{-1}$ for some $r \in R, s \in S$. More colloquially, elements of R_S look like "fractions" $\frac{r}{s}$ with $r \in R, s \in S$.

(B) The kernel of ϵ_S is the elements annihilated by some element of S.

$$\ker \epsilon_S = \{r \in R : sr = 0 \text{ for some } s \in S\}$$

Example 4.1. Let $R = \mathbb{Z}$ and let $S = \mathbb{Z} \setminus \{0\}$. The localization R_S is $\mathbb{Q}$. More generally, if R is any integral domain and S is the set of all nonzero elements, then R_S is the ring of fractions of R, which is a field. The localization map $\epsilon_S : R \to R_S$ is the "inclusion" which just sends an element x to the fraction $\frac{x}{1}$. The kernel is trivial as R is an integral domain.

4.1.2 Possibly noncommutative rings

We want to know what happens when R is not necessarily commutative. Can we still form a localization R_S? Will it still have the properties above? If not, what conditions can we impose on S in order to get properties like this?

We will show that R_S still exists, the universal property still holds, but properties (A), (B) above may fail. Basically, this means that R_S is very unmanageable in this fully general situation, since we can't even think of the elements as fractions. The remedy for this will be to impose conditions in S, to get some more control on R_S.

Proposition 4.2. *Let R be an associative unital (not necessarily commutative) ring and let $S \subset R \setminus \{0\}$ be a multiplicative subset. Then there exists an associative unital ring R_S and a morphism of rings $\epsilon_S : R \to R_S$ such that*

1. *$\epsilon_S(s) \in R_S^\times$ for all $s \in S$*
2. *ϵ_S is universal with this property.*

Proof. To each $r \in R$, we associate a symbol x_r, and to each $s \in S$, we associate a symbol y_s. So each $s \in S$ has two associated symbols, x_s and y_s which are distinct. Set

$$X = \{x_r : r \in R\}$$
$$Y = \{y_s : s \in S\}$$
$$T = X \cup Y$$

Let $\mathbb{Z}\langle T\rangle$ be the free $\mathbb{Z}$-algebra on T. Informally, this is a polynomial ring over $\mathbb{Z}$ in non-commuting variables x_r and y_s. Let $I \subset \mathbb{Z}\langle T\rangle$ be the 2-sided ideal generated by all relations in R, together with

$$x_s y_s - 1 \qquad y_s x_s - 1$$

for all $s \in S$. By "all relations in R" we mean that if $a + b = c$ in R, then $a + b - c = 0$ so we take $a + b - c$ as a generator of I, for example. Then define

$$R_S := \mathbb{Z}\langle T\rangle / I$$

and define

$$\epsilon_S : R \to R_S \qquad r \mapsto \overline{x}_r = x_r + I$$

This is a ring homomorphism because I contains all additive and multiplicative relations in R by construction. It is also immediate from the construction that $\epsilon_S(s) = \overline{x}_s$ is a unit, since $\overline{y}_s$ is the 2-sided inverse.

$$\overline{x}_s \overline{y}_s = \overline{x_s y_s} = \overline{1}$$

since $x_s y_s - 1 \in I$. All that remains is to verify the universal property. Suppose $\theta : R \to R'$ is a ring homomorphism such that $\theta(s) \in (R')^\times$ for all $s \in S$. Then there exists $t_s \in R'$ such that

$$\theta(s) t_s = t_s \theta(s) = 1$$

In other words, $t_s = \theta(s)^{-1}$. Define

$$\widetilde{\omega} : \mathbb{Z}\langle T\rangle \to R' \qquad x_r \mapsto \theta(r) \qquad y_s \mapsto t_s$$

Since the above defines $\widetilde{\omega}$ only on the generating set, it extends uniquely to a ring homomorphism. The map $\widetilde{\omega}$ vanishes on the generators $x_s y_s - 1$ and $y_s x_s - 1$ of I by definition of t_s.

$$\widetilde{\omega}(x_s y_s - 1) = \theta(s) t_s - 1 = 1 - 1 = 0$$

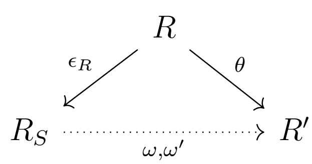
It vanishes on other generators of I because θ is a ring homomorphism. For example, if $a + b = c$ in R, then

$$\widetilde{\omega}(x_a + x_b - x_c) = \theta(a) + \theta(b) - \theta(c) = \theta(a + b - c) = \theta(0) = 0$$

So $\widetilde{\omega}$ induces a ring homomorphism

$$\omega : R_S \to R \qquad \overline{x}_r \mapsto \theta(r) \qquad \overline{y}_s \mapsto t_s$$

which makes the required diagram commute for the universal property. Then we should verify that ω is unique. Suppose $\omega' : R_S \to R'$ also makes the diagram commute.



By the diagram, $\omega'(\overline{x}_r) = \theta(r)$ for all $r \in R$, and

$$\theta(s)\omega'(\overline{y}_s) = \omega'(\overline{x}_s)\omega'(\overline{y}_s) = \omega'(\overline{x}_s \overline{y}_s)\omega'(\overline{1}) = 1$$

Similarly, $\omega'(\overline{y}_s)\theta(s) = 1$. Thus $\omega'(\overline{y}_s)$ is the 2-sided inverse of $\theta(s)$. But multiplicative 2-sided inverses in arbitrary associative rings are still unique, and t_s is also a 2-sided inverse for $\theta(s)$, so $\omega'(\overline{y}_s) = t_s$. Thus ω' agrees with ω on the generators $\overline{x}_r, \overline{y}_s$ for all $r \in R, s \in S$, so $\omega' = \omega$. □

4.1.3 Pathologies in localizing noncommutative rings

The localization constructed in the level of generality above can behave somewhat strangely. We give an example below.

Example 4.3 (Failure of property (B))**.** Recall that in the commutative case, we can characterize the kernel of the canonical map ϵ_S as annihilators of elements of S, which we called property (B).

$$\text{(B)} \qquad \ker \epsilon_S = \{r \in R : sr = 0 \text{ for some } s \in S\}$$

We show this fails in the noncommutative case. Fix a field K and an integer $n \geq 2$, and let $R = \mathrm{M}_n(K)$. For a multiplicative subset, let $S = \{1, E_{11}\}$. This is multiplicative because $E_{11}^2 = E_{11}$. By the proposition, we have a localization R_S and map $\epsilon_S : R \to R_S$ which makes $\epsilon_S(E_{11})$ into a unit. However, in R, E_{11} is a zero divisor. In particular,

$$E_{11}E_{22} = 0 \implies \epsilon_S(E_{11}E_{22}) = \epsilon_S(E_{11})\epsilon_S(E_{22}) = 0$$

Since $\epsilon_S(E_{11})$ is a unit, this implies $\epsilon_S(E_{22}) = 0$. In particular, the kernel of ϵ_S is nontrivial. Since $R = M_n(K)$ is a simple ring, it has nontrivial 2-sided ideals, and $\ker \epsilon_S$ is such an ideal, so $\ker \epsilon_S = R$, and ϵ_S is actually the zero map. This forces the localization R_S to be the zero ring.

But more importantly, $\ker_S$ is not just elements of R which annihilate elements of S, it is larger than this. For example, $E_{11} \in R$ does not annihilate any element of S, but is killed by ϵ_S.

Remark 4.4. The only important properties in the previous example were that R was non-commutative, and the set S contained a zero divisor (in the example, E_{11} was our zero divisor), and that the annihilator of that zero divisor is not the entire ring R. So many other similar examples exist.

The previous example shows that property (B) fails, but in that example property (A) is vacuously satisfied, since the localization R_S was just the zero ring. Counterexamples for property (A) are more complicated, but we sketch one. First we need to discuss division hulls.

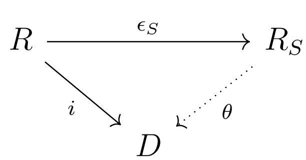
Definition 4.5. Let R be a ring and D a division ring. An ring homomorphism $i : R \hookrightarrow D$ is a **division hull** if it is injective, and no proper division subring of D contains $i(R)$.

Lemma 4.6. *Let R be an associative unital ring with no zero divisors, and let $S = R \setminus \{0\}$ be the multiplicative subset of all nonzero elements. Assume that every element of R_S can be written in the form $\epsilon_S(r)\epsilon_S(s)^{-1}$ for some $r \in R, s \in S$.*

1. *Then $\epsilon_S : R \to R_S$ is a division hull.*
2. *If $i : R \hookrightarrow D$ is a division hull of R, then $D \cong R_S$, so R has a unique division hull (up to isomorphism).*

Proof. (1) Omitted.

(2) Since i maps elements of S to units in D, by the universal property, there exists a unique morphism $\theta : R_S \to D$ making the following diagram commute.



Since $\theta(R_S)$ is a division ring containing $i(R)$, and D is a division hull, we must have $\theta(R_S) = D$. Since θ is a morphism of division rings and is surjective, it must also be injective, so θ is an isomorphism between D and R_S. $\square$

Example 4.7 (Failure of property (A))**.** We give an example of a noncommutative ring R and multiplicative subset S so that the localization R_S fails property (A), that is, not every element can be written in the form $\epsilon_S(r)\epsilon_S(s)^{-1}$. Let K be a field and let

$$R = K \langle u, v \rangle$$

be the polynomial ring in noncommuting variables over K, and let $S = R\backslash\{0\}$ be all nonzero elements. Note that R has no zero divisors[9]. One can show that for $n \in \mathbb{Z}_{\geq 1}$, there exist division hulls

$$\epsilon_n : R \to D_n$$

and the division rings D_n have the property that if $n \neq m$, then there are no ring homomorphisms $D_n \to D_m$. In particular, this shows that R has multiple division hulls which are NOT isomorphic. By lemma 4.6, if R_S has property (A), then it would have a unique (up to isomorphism) division hull. Since it does not, it must not have property (A). For details on the construction of these strange division hulls, see *Lectures on Modules and Rings* by Lam, Theorem 9.27, on page 296.

4.1.4 Denominator sets and Ore conditions

Next up we seek conditions to impose on S which give us some control over the localization R_S, since the previous examples illustrate how R_S can be quite strange in the full generality of associative unital rings.

Definition 4.8. Let R be an associative unital ring, and $S \subset R\backslash\{0\}$ a multiplicative subset. A **right ring of fractions of *R* with respect to *S*** is a ring R' with a ring homomorphism $\phi : R \to R'$ such that

(a) $\phi(s) \in (R')^\times$ for all $s \in S$.

(b) Every element of R' has the form $\phi(r)\phi(s)^{-1}$ for some $r \in R, s \in S$.

(c) The kernel of ϕ is precisely annihilators of elements of S.

$$\ker \phi = \{r \in R : rs = 0 \text{ for some } s \in S\}$$

As our examples have demonstrated, a noncommutative localization R_S need not be a right ring of fractions. As a side note, one can define a left ring of fractions by reversing the multiplications in (2) and (3), but we won't make use of this dual notion.

Lemma 4.9 (Necessary conditions for a right ring of fractions)**.** *Suppose S, R are as above, and $\phi : R \to R'$ is a right ring of fractions with respect to S. Then*

(I) $aS \cap sR \neq \emptyset$ for all $a \in R, s \in S$

(II) If $a \in R, s \in S$, and $sa = 0$, then there exists $t \in S$ such that $at = 0$.

Proof. (I) Let $a \in R, s \in S$. By property (b) of a right ring of fractions, $\phi(s)^{-1}\phi(a)$ can be written as $\phi(r)\phi(t)^{-1}$ for some $r \in R, t \in S$.

$$\phi(s)^{-1}\phi(a) = \phi(r)\phi(t)^{-1}$$

[9] Perhaps it is necessary to assume that K has characteristic zero to know that R has no zero divisors, I am not sure.

We can multiply both sides by $\phi(t)$ on the right and $\phi(s)$ on the left to get

$$\phi(a)\phi(t) = \phi(s)\phi(r) \implies \phi(at) = \phi(sr) \implies \phi(at - sr) = 0$$

So $at - sr \in \ker \phi$. Then by property (c), there exists $u \in S$ such that

$$(at - sr)u = 0 \implies atu = sru$$

Clearly $atu \in aR$ and $srU \in sR$, so the element above lies in the intersection $aS \cap sR$.

(II) Suppose $a \in R, s \in S, sa = 0$. Then $\phi(s)\phi(a) = 0$ implies $\phi(a) = 0$ since $\phi(s)$ is a unit. By property (c), there exists $t \in S$ such that $at = 0$. □

Definition 4.10. Condition (I) in the previous lemma is called the **right Ore condition**, and a subset $S \subset R \setminus \{0\}$ satisfying it is called a **right Ore subset** of R.

Definition 4.11. Let R be an associative unital ring and $S \subset R \setminus \{0\}$ a multiplicative subset. S is a **right denominator set** for R if it satisfies properties (I) and (II) from the previous proposition.

Theorem 4.12. *Let R be an associative unital ring and $S \subset R \setminus \{0\}$ a multiplicative subset. Then R has a right ring of fractions with respect to S if and only if S is a right denominator set.*

Proof. Lemma 4.9 shows that if R has a right ring of fractions with respect to S, then S is a right denominator set. We sketch the converse argument. Suppose S is a right denominator set. Define a relation $\sim$ on $R \times S$ by

$$(a, s) \sim (a', s") \iff \exists b, b \in R \text{ such that } sb = s'b' \text{ and } ab = a'b'$$

Informally, we think of (a, s) as a "fraction" $\frac{a}{s}$. Most of the proof is a lengthy verication that $\sim$ is an equivalence relation on $R \times S$. In particular, transitivity is where conditions (I) and (II) are used. After showing this, define

$$R' := (R \times S)/\sim$$

The equivalence class of (a, s) is denoted $\frac{a}{s}$. Then define

$$\phi : R \to R' \qquad a \mapsto \frac{a}{1}$$

Then define addition in R' as follows. Suppose we have $\frac{a_1}{s_1}, \frac{a_2}{s_2} \in R'$. By condition (I), $s_1 S \cap s_2 R \neq \emptyset$, so there exist $r \in R, s \in S$ such that $s_2 r = s_1 s = t \in S$, and by definition of the equivalence,

$$\frac{a_1}{s_1} = \frac{a_1 s}{s_1 s} \qquad \frac{a_2}{s_2} = \frac{a_2 r}{s_2 r}$$

And now the versions on the right sides of the respective equalities have the same "denominator," so we can add them in the usual way.

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} := \frac{a_1 s + a_2 r}{t}$$

where $t = s_2 r = s_1 s$. Then one checks that this addition is well-defined, and is is not too hard to verify that ϕ is additive. It is also easy to see that

$$\ker \phi = \{r \in R : rs = 0 \text{ for some } s \in S\}$$

Multiplication in R' is defined as follows. Given fractions $\frac{a_1}{s_1}, \frac{a_2}{s_2}$, by (I) $a_2 S \cap s_1 R \neq \emptyset$, so there exists $r \in R$ and $s \in S$ so that $s_1 r = a_2 s$. Then we define[10]

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} := \frac{a_1 r}{s_2 s}$$

Then one checks that the multiplication is is well defined, with identity $\frac{1}{1}$, and that ϕ is multiplicative, so ϕ is a ring homomorphism. □

Remark 4.13. The ring R' from the previous proof is usually denoted RS^{-1}.

Corollary 4.14. *If S is a right denominator set, then $\phi : R \to RS^{-1}$ is the localization of R at S. In particular, there exists a unique isomorphism $\omega : R_S \to RS^{-1}$ such that $\omega \epsilon_S = \phi$.*

Proof. Just show that $\phi : R \to RS^{-1}$ satisfies the same universal property as ϵ_S. Details skipped. □

4.2 Localization of arbitrary categories

Localization in categories bears a lot of resemblances to localizing an arbitrary associative unital ring.

4.2.1 General localization

We start out with a very general construction - given any category (no assumptions about being additive, abelian, or triangulated), and any class of morphisms, we can "'formally invert" those morphisms to form a new category in which those morphisms are isomorphisms. We also get a morphism from our original category to the localized category, paralleling the localization morphism from a ring to its localization.

However, as we'll soon see, this general localization construction fails to have many useful and desirable properties, so we'll soon abandon it in favor of another localization construction which depends upon imposing categorical analog of properties of a right denominator set, in order to have a localization construction with a categorical analog of properties of a right ring of fractions.

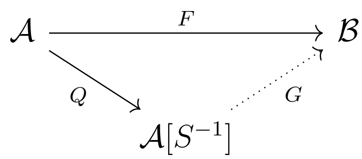
Theorem 4.15. *Let $\mathcal{A}$ be a category, and S any class of morphisms in $\mathcal{A}$. Then there exists a category $\mathcal{A}[S^{-1}]$ and a functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ such that*

1. *$Q(s)$ is an isomorphism for all $s \in S$*

[10] The intuition behind this definition is that

$$(a_1 s_1^{-1})(a_2 s_2^{-1}) = a_1(s_1^{-1} a_2) s_2^{-1} = a_1(rs^{-1}) s_2^{-1} = (a_1 r)(s^{-1} s_2^{-1}) = (a_1 r)(s_2 s)^{-1}$$

2. Q is universal with this property. More precisely, if $F : \mathcal{A} \to \mathcal{B}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$, then there exists a unique functor $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that the following diagram commutes.



Remark 4.16. As usual, a universal property such as the (2) above means that the category $\mathcal{A}[S^{-1}]$ is unique up to isomorphism. However, now that we're working with categories, isomorphism is a much stronger notion than the analogous condition of isomorphism in the category of rings. Usually we consider categories basically "the same" if they are just equivalent, which is significantly weaker than isomorphism of categories. Just something to keep in mind.

Proof. The objects of $\mathcal{A}[S^{-1}]$ are just the objects of $\mathcal{A}$. For the morphisms, we do something more convoluted. Given two objects X, Y in $\mathcal{A}$, define a **directed edge** between them to be either a homomorphism $f : X \to Y$ with $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, or for $s \in S \cap \mathrm{Hom}_{\mathcal{A}}(X, Y)$, we have a directed edge $Y \xleftarrow{s} X$. Then define a **path** between two objects M, N in $\mathcal{A}$ to be a sequence of objects $L_0 = M, L_1, L_2, \dots, L_n = N$ connected by directed edges. Note that for $s \in S \cap \mathrm{Hom}_{\mathcal{A}}(X, Y)$, we have two distinct directed edges

$$X \xrightarrow{s} Y \qquad Y \xleftarrow{s} X$$

Next we define four elementary transformations of paths, which we depict in the table.

Original	Replacement	Requirements
$X \xrightarrow{f} Y \xrightarrow{g} Z$	$X \xrightarrow{gf} Z$	None
$X \xrightarrow{s} Y \xleftarrow{s} X$	$X \xrightarrow{\mathrm{Id}_X} X$	$s \in S$
$Y \xleftarrow{s} X \xrightarrow{s} Y$	$Y \xrightarrow{\mathrm{Id}_Y} Y$	$s \in S$
$X \xrightarrow{\mathrm{Id}_X} X \xleftarrow{s} Y$	$X \xleftarrow{s} Y$	$s \in S$

Finally, we define two paths between objects M, N to be **equivalent** if they can be transformed into each other using the four elementary transformations above. We omit the details, but this is an equivalence relation. Then define $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ to be the equivalence classes of paths under this relation.[11] Composition in $\mathcal{A}[S^{-1}]$ is by concatenation of paths, and the identity morphism is the equivalence class of the path of length one consisting of the identity morphism from $\mathcal{A}$. It's clear that composition is associative.

Next we define the functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$. On objects, it is just $Q(M) = M$. On a morphism $f : M \to N$ in $\mathcal{A}$, Q sends it to the equivalence class of that path of length one

[11] There is a minor set-theoretic issue here where in an arbitrary $\mathcal{A}$, even if $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a set for any given objects X, Y, the paths from X to Y may be a proper class, if the collection of objects in $\mathcal{A}$ is a proper class. So to do all this one needs to work out whether it makes sense to have equivalence classes in proper classes, but this is beyond the scope of these notes.

consisting of f. Let $s : M \to N$ be in S. Then $Q(s)$ is invertible, with inverse given by the equivalence class of the path $N \stackrel{s}{\leftarrow} M$.

It just remains to check that Q satisfies the universal property. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a functor such that $F(s)$ is an isomorphism for all $s \in S$. Define $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ as follows. On objects, $G(M) = F(M)$. Given a morphism $\phi : M \to N$ in $\mathcal{A}[S^{-1}]$, choose a path representing it. For directed edges which are morphisms in $\mathcal{A}$, denote them by $L_i \xrightarrow{f_i} L_{i+1}$ and for directed edges coming from elements of S, denote them by $L_i \xleftarrow{t_i} L_{i+1}$.

$$M = L_0 \xrightarrow{f_0} L_1 \xleftarrow{t_1} L_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} L_n = N$$

Each t_i is a morphism $L_i \to L_{i+1}$ in $\mathcal{A}$, and $F(t_i)$ is an isomorphism in $\mathcal{B}$. We denote the inverse by $F(t_i)^{-1}$. Applying F to each f_i and taking the inverse for each t_i, we obtain a path (consisting of actual morphisms) in B.

$$F(M) = F(L_0) \xrightarrow{F(f_0)} F(L_1) \xrightarrow{F(t_1)^{-1}} F(L_2) \xrightarrow{F(f_2)} \dots \xrightarrow{F(f_n)} F(L_n) = F(N)$$

We define $G(\phi)$ to be the composition along this path in $\mathcal{B}$. That is,

$$G(\phi) = F(f_n) \circ F(f_{n-1}) \circ \cdots \circ F(f_2) \circ F(t_1)^{-1} \circ F(f_0)$$

It is somewhat tedious to verify that this definition does not depend on the choice of path representing ϕ, this involves working with the elementary transformations, so we omit the verification. After this is verified, we have a well defined assignment

$$G : \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(H, M) \to \mathrm{Hom}_{\mathcal{B}}(G(M), G(N))$$

making G a functor. Lastly, it is immediate from the definition of G that $G \circ Q = F$, but we spell it out. Given a morphism $M \xrightarrow{f} N$ in $\mathcal{A}$, the image under Q is the equivalence class of the path of length one consisting of f. By definition of G, applying G to the equivalence class is just choosing a representive and applying F to each edge. We can just choose $M \xrightarrow{f} N$ as our represenative, so then $G \circ Q(f) = F(f)$.

Lastly, we need to check that G is unique. It suffices to check that G is uniquely determined on edges, but this is immediate from the commutative diagram which G has to satisfy, and the fact that even in arbitrary categories, 2-sided inverses are unique. □

Definition 4.17. The functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ in the theorem is called the **localization functor**.

Remark 4.18. The previous construction is as general as possible. We put no restrictions on the category $\mathcal{A}$, or on the collection of morphisms to invert. However, the resulting path category $\mathcal{A}[S^{-1}]$ is incredibly unwieldy to work with, so it is necessary to understand what additional conditions on S give some more control of $\mathcal{A}[S^{-1}]$.

4.2.2 Localizing classes

Next up we define our categorical analog of right denominator sets. The first few properties are just categorical versions of being a multiplicative subset, then we get to the right denominator set properties.

Definition 4.19. Let $\mathcal{A}$ be a category and S a class of morphisms in $\mathcal{A}$. S is a **localizing class** if it satisfies

(LC1) For all objects M of $\mathcal{A}$, $\mathrm{Id}_M \in S$.

(LC2) If $s, t \in S$ can be composed, then their composition is in S.

(LC3a) For any morphisms $M \xrightarrow{f} N$ and $L \xrightarrow{s} N$ with $s \in S$, there exist morphisms $K \xrightarrow{g} L$ and $K \xrightarrow{t} M$ with $t \in S$ such that the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{g} & L \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow[f]{} & N \end{array}$$

(LC3b) For any morphisms $N \xrightarrow{f} M$ and $N \xrightarrow{s} L$ with $s \in S$, there exist morphisms $L \xrightarrow{g} K$ and $M \xrightarrow{t} K$ with $t \in S$ such that the following diagram commutes.

$$\begin{array}{ccc} K & \xleftarrow{g} & L \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} \\ M & \xleftarrow{f} & N \end{array}$$

(LC4) For any morphisms $f, g : M \to N$,

$$\exists s \in S, sf = sg \iff \exists t \in S, ft = gt$$

Remark 4.20. We give some intuition behind the conditions (LC1)-(LC4). Conditions (LC1) and (LC2) are analogous to saying that S is a multiplicative subset of a ring. (LC3) is analogous to the right Ore condition $aS \cap sR \neq \emptyset$, and (LC3b) is analogous to the left Ore condition. (LC4) is analogous to condition (II) for rings, which said

$$s \in S, at = 0 \implies \exists t \in S, at = 0$$

In fact, if $\mathcal{A}$ is an additive category, then (LC4) is equivalent to something looking exactly like condition (II). So we think of conditions (LC1)-(LC4) as saying that S is a left and right denominator set, in a categorical sense.

Remark 4.21. It is possible to think of a ring as a category, and make the previous analogies more precise. Let R be an associative unital (not necessarily commutative) ring. Associated to R is a category $\mathcal{R}$ with one object $*$, and morphisms given by elements of R. Composition of morphisms is multiplication in R, and $\mathrm{Hom}_{\mathcal{R}}(*, *)$ is an abelian group under addition in R.

Let S be a set of morphisms in $\mathcal{R}$, a.k.a. a subset of R. Then condition (LC1) is equivalent to S containing 1_R, and condition (LC2) is equivalent to S being closed under multiplication. I don't have the patience to work it out, but presumably condition (LC3a) is equivalent to the right Ore condition, (LC3b) is equivalent to the left Ore condition, and (LC4) is equivalent to property (II).

Our next objective is to prove the following.

Proposition 4.22. *Let S be a localizing class in a category $\mathcal{A}$ and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ the localization functor. Every morphism in $\mathcal{A}[S^{-1}]$ can be represented in the form $Q(f) \circ Q(s)^{-1}$ with $s \in S$, and also in the form $Q(t)^{-1} \circ Q(g)$ with $t \in S$.*

First, a remark and then a lemma.

Remark 4.23. Let S be a localizing class, let $s : Y \to Z$ and $t : X \to Y$ be in S, and consider the morphism $\phi : Z \to X$ in $\mathcal{A}[S^{-1}]$ which is the equivalence class of the path

$$\cdot \xleftarrow{s} \cdot \xleftarrow{t} \cdot$$

We use dots to represent all the objects, because they aren't important, other than the fact that we can compose s, t. Using elemenetary transformations, both of the following paths reduce to the identity.

$$\cdot \xrightarrow{t} \cdot \xrightarrow{s} \cdot \xleftarrow{s} \cdot \xleftarrow{t} \cdot$$
$$\cdot \xleftarrow{s} \cdot \xleftarrow{t} \cdot \xrightarrow{t} \cdot \xrightarrow{s} \cdot$$

Thus the equivalence class of the path

$$\cdot \xrightarrow{t} \cdot \xrightarrow{s} \cdot$$

is inverse to the morphism ϕ in $\mathcal{A}[S^{-1}]$. But again by the elementary transformations, this path representing ϕ^{-1} is equivalent to the path

$$\cdot \xrightarrow{s \circ t} \cdot$$

By (LC2), $s \circ t \in S$, so we have a path

$$\cdot \xleftarrow{s \circ t} \cdot$$

which clearly represents the inverse of ϕ^{-1}. That is to say, ϕ is also represented by the path

$$\cdot \xleftarrow{s \circ t} \cdot$$

To summarize, if $s, t \in S$ can be composed appropriately, then the elementary transformations allow the substitution

$$\cdot \xleftarrow{s} \cdot \xleftarrow{t} \cdot \qquad \rightsquigarrow \qquad \cdot \xleftarrow{s \circ t} \cdot$$

Lemma 4.24. *Let S be a localizing class in a category $\mathcal{A}$ and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ the localizing functor. Every morphism in $\mathcal{A}[S^{-1}]$ can be written in the form*

$$\left(Q(f_1) \circ Q(s_1)^{-1}\right) \circ \left(Q(f_2) \circ Q(s_2)^{-1}\right) \circ \cdots \circ \left(Q(f_n) \circ Q(s_n)^{-1}\right)$$

for some morphisms $f_1, \ldots, f_n$ from $\mathcal{A}$ and morphisms $s_1, \ldots, s_n \in S$.

Proof. We start with an arbitrary morphism $g : M \to N$ in $\mathcal{A}[S^{-1}]$, and choose a path representing it.

$$M = L_0 \xrightarrow{g_1} L_1 \to \cdots \to L_n = N$$

The picture above is somewhat deceptive because we don't know yet that the morphisms are rightward oriented, but it's just a picture. By adding either $L_0 \xrightarrow{\text{Id}} L_0$ or $L_0 \xrightarrow{\text{Id}} L_0 \xleftarrow{\text{Id}} L_0$ to the start, we can assume that the starting edge is rightward oriented, and similarly by adding $L_n \xleftarrow{\text{Id}} L_n$ or $L_n \xrightarrow{\text{Id}} L_n \xleftarrow{\text{Id}} L_n$ to the end, we can sassume the path ends with a leftward oriented edge.

Next, any two consecutive rightward oriented edges can be replaced by their composition, using an elementary transformation. Similarly, if we have two consecutive leftward oriented edges $L_i \xleftarrow{s_i} L_{i+1} \xleftarrow{s_{i+1}} L_{i+2}$, with $s_i, s_{i+1} \in S$, then by remark 4.23, we can use the elementary transformations to substitute a single leftward oriented edge $L_i \xleftarrow{s_{i+1} \circ s_i} L_{i+2}$.

So we can combine any consecutive edges in the same direction, and force our path to start with a forward edge and end with a backward edge. So our path has the form

$$\cdot \to \cdot \leftarrow \cdot \to \cdot \leftarrow \cdot \to \cdots \to \cdot \leftarrow \cdot$$

Then we apply Q to this path, and we the same morphism in $\mathcal{A}[S^{-1}]$, since we only modified the path representing it using elementary transformations. □

Now we can prove proposition 4.22, which we restate for convenience.

Proposition 4.25. *Let S be a localizing class in a category $\mathcal{A}$ and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ the localization functor. Every morphism in $\mathcal{A}[S^{-1}]$ can be represented in the form $Q(f) \circ Q(s)^{-1}$ with $s \in S$, and also in the form $Q(t)^{-1} \circ Q(g)$ with $t \in S$.*

Proof. By lemma 4.24, we can write a morphism in the form

$$\Big(Q(f_1) \circ Q(s_1)^{-1}\Big) \circ \cdots \circ \Big(Q(f_n) \circ Q(s_n)^{-1}\Big)$$

If we can show that given such a representation, we can always reduce the length and express it in the same form with $< n$ terms, then by induction we can write every morphism as $Q(f) \circ Q(s)^{-1}$. Assume $n > 1$. Then consider the first few parts of the path.

$$Q(f_1) \circ Q(s_1)^{-1} \circ Q(f_2) \circ Q(s_2)^{-1}$$

By (LC3a), there exists morphisms g, t with $t \in S$ making the following diagram commute.

$$\begin{array}{ccc} K & \overset{g}{\dashrightarrow} & L \\ {\scriptstyle t}\,\vdots & & \downarrow{\scriptstyle s_1} \\ M & \xrightarrow{f_2} & N \end{array}$$

So $s_1 g = f_2 t$, and applying Q and doing some rearranging, $Q(s_1)^{-1}Q(f_2) = Q(g)Q(t)^{-1}$. So

$$Q(f_1) \circ Q(s_1)^{-1} \circ Q(f_2) \circ Q(s_2)^{-1} = Q(f_1) \circ Q(g)Q(t)^{-1} \circ Q(s_2)^{-1} = Q(f_1 g) \circ Q(s_2 t)^{-1}$$

So we can always reduce the length, completing the induction. Regarding the statement of the proposition regarding writing any morphism in the form $Q(t)^{-1}Q(g)^{-1}$, the argument is exactly analogous, but requires (LC3b) for the induction step. □

Remark 4.26. As far as I can tell, the preceding proposition doesn't depend on property (LC4) directly or indirectly, so it should hold even if S is just a class satisfying (LC1)-(LC3). This probably never matters, but you never know.

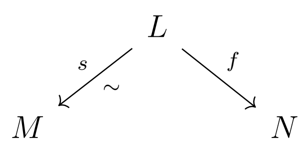
Remark 4.27. Regarding our eventual goal of defining the derived category, we will eventually show that if $\mathcal{A}$ is an abelian category with homotopy category $K(\mathcal{A})$ and S is the class of quasi-isomorphisms in $K(\mathcal{A})$, then

1. S is a localizating class, so
2. We can define the derived category of $\mathcal{A}$ as $D(\mathcal{A}) := K(\mathcal{A})[S^{-1}]$.
3. The derived category $D(\mathcal{A})$ inherits a triangulated structure from $K(\mathcal{A})$.

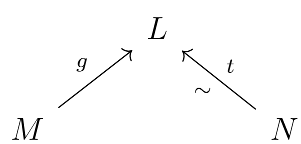
4.2.3 Roofs

We continue our discussion of localized categories $\mathcal{A}[S^{-1}]$ when S is a localizing class, but develop some terminology of roofs to describe the morphisms in $\mathcal{A}[S^{-1}]$ in a different way.

Definition 4.28. Let $\mathcal{A}$ be a category and S a localizing class of morphisms in $\mathcal{A}$. For $s \in S$, the morphism $Q(f) \circ Q(s)^{-1}$ in $\mathcal{A}[S^{-1}]$ is represented by a **left roof**, which is a diagram



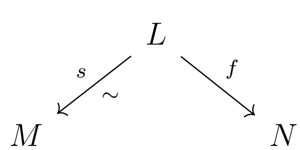
The roof above is **from** M **to** N. Similarly, $Q(t)^{-1} \circ Q(g)$ with $t \in S$ is represented by a **right roof** which looks like

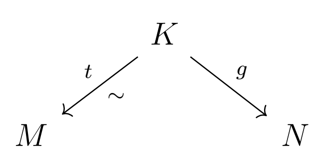


The $\sim$ symbols are just used to indicate elements of S, and remind us that in $\mathcal{A}[S^{-1}]$ they are isomorphisms.

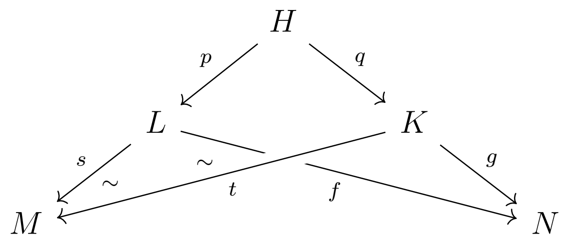
Remark 4.29. We reinterpret proposition 4.25 as saying that every morphism in $\mathcal{A}[S^{-1}]$ can be represented by a right roof and by a left roof. However, the question still remains: when do two left roofs represent the same morphism? We answer this eventually in lemma 4.44.

Definition 4.30. Suppose we have two left roofs both from M to N.





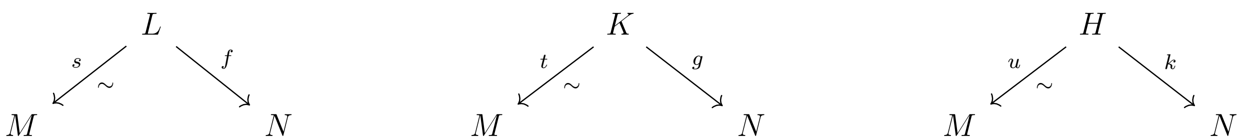
We say these left roofs are **equivalent** if there exist morphisms $p : H \to L$ and $q : H \to K$ such that $sp, tq \in S$ and making the following diagram commute.[12]



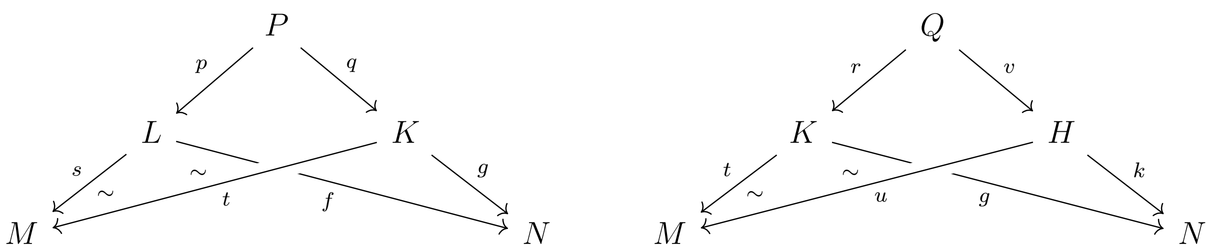
Remark 4.31. Equivalence of right roofs is defined analogously.

Lemma 4.32. *Equivalence of left roofs is an equivalence relation.*

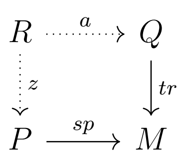
Proof. Reflexivity and symmetry are easy. We prove transitivity. Suppose we have three left roofs from M to N, where the first two are equivalent and the second two are equivalent.



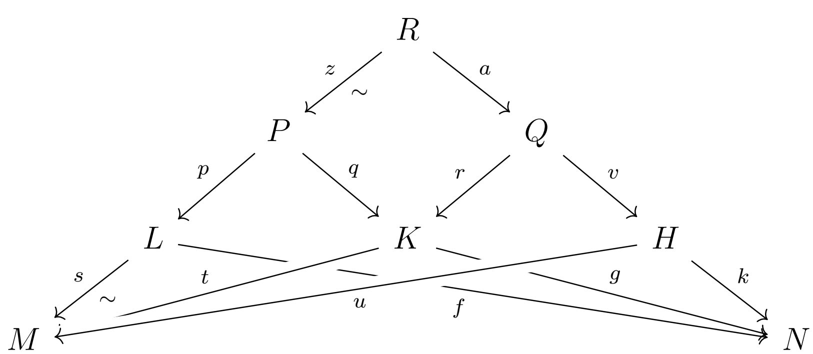
From the equivalences, we get the following commutative diagrams.



with $sp = tq, tr = uv \in S$. By (LC3a), there is a commutative diagram

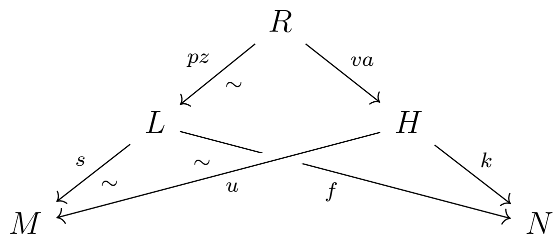


with $z \in S$. This fits into a giant diagram



[12]The diagram says $sp = tq$ so the requirement that both lie in S is redundant.

Be careful with this diagram. Everything commutes except possibly the top square involving R, P, Q, K commutes; we do NOT know that $ra = qz$, but it turns out this is not necessary. Since $sp \in S$ and $z \in S$ by (LC2) $spz = tra = uva \in S$. Condensing the diagram a bit, we have



This gives the required equivalence between the two roofs

∎

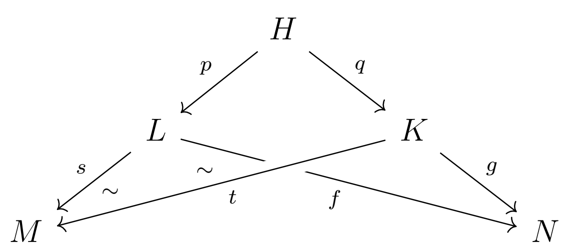
Remark 4.33. I didn't use (LC4) in the previous argument, but when my professor did it he used (LC4) somewhere, but I didn't quite understand why it was needed. Perhaps my argument is wrong and the fix requires (LC4), I wouldn't be surprised.

Remark 4.34. A similar argument shows that the analogous relation on right roofs is also an equivalence relation.

Lemma 4.35. *If two left roofs are equivalent, then they represent the same morphism in* $\mathcal{A}[S^{-1}]$.

Proof. Consider the roofs

If they are equivalent, then we have our commutative diagram

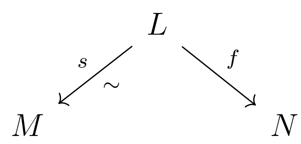


with $sp \in S$. So $Q(sp) = Q(s)Q(p)$ is an isomorphism (in $\mathcal{A}[S^{-1}]$). Since $s \in S$, $Q(s)$ is an isomorphism, so $Q(p)$ is also an isomorphism. Similarly, $Q(q)$ is an isomorphism. So

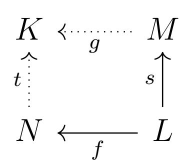
$$Q(f)Q(s)^{-1} = Q(fp)Q(sp)^{-1} = Q(gq)Q(tq)^{-1} = Q(g)q(t)^{-1} = Q(g)Q(t)^{-1}$$

The above is an equality of morphisms in $\mathcal{A}[S^{-1}]$, between the morphisms represented by our two roofs. $\square$

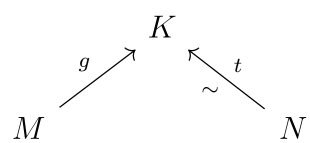
Remark 4.36. We now describe a bijection between equivalence classes of left roofs and equivalence classes of right roofs. We start with a left roof.



By (LC3b), there exist morphisms t, g with $t \in S$ fitting into the following commutative square.

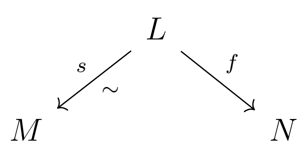


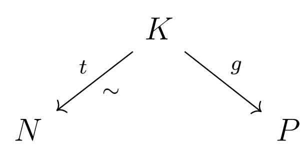
Since $t \in S$, the following is a right roof.



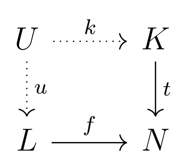
This associates a left roof to a right roof. There is a great deal of checking involved, but the association of the equivalence class of the first roof above to the equivalence class of the second roof is a bijection. We omit the details. The main purpose of this is to say that working with left roofs exclusively loses no generality.

Definition 4.37 (Composition of roofs)**.** We now define composition of left roofs. Suppose we have two left roofs, from M to N and N to P respectively.

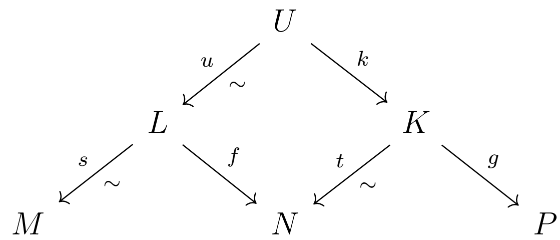




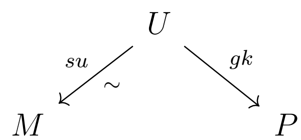
By (LC3a), there is a commutative diagram



with $u \in S$. This fits into the following commutative diagram.



By (LC2) $su \in S$, so we have a left roof from M to P.



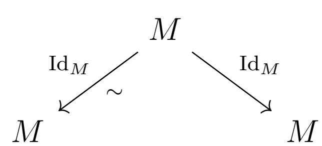
We define this to be the **composition** of the two roofs we started with.

Lemma 4.38. *Composition of left roofs has the following properties.*

1. *Composition of left roofs descends to equivalence classes. That is, if x, x' are equivalent roofs and y, y' are equivalent roofs, then the composition $x \circ y$ is equivalent to the composition of $x' \circ y'$. (Assuming x, y have domain/codomain such that the composition $x \circ y$ makes sense.)*

2. *Composition is associative.*

3. *Composition has an identity given by the (equivalence class of) the left roof*

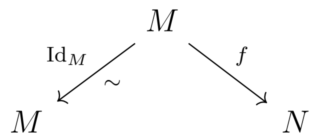


In summary, if S is a localizing class of morphisms in a category $\mathcal{A}$, then there is a category $\mathcal{A}_S^\ell$ whose objects are objects of $\mathcal{A}$, and whose morphisms are equivalence classes of left roofs with composition define as above.

Proof. Very boring details. □

Remark 4.39. Composition is defined for right roofs analogously with that of left roofs, and there is similarly a category $\mathcal{A}_S^r$ with morphisms given by equivalence classes of right roofs. Additionally, the bijection between equivalence classes of left and right roofs preserves the respective compositions, which induces an equivalence of categories $\mathcal{A}_S^\ell \cong \mathcal{A}_S^r$. So we just denote the category of left roofs by $\mathcal{A}_S$.

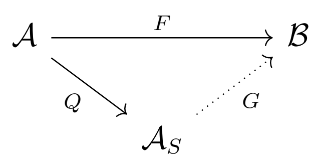
Definition 4.40. We define a functor $Q : \mathcal{A} \to \mathcal{A}_S$ as follows. On morphisms Q is the identity. On a morphism $f : M \to N$ in $\mathcal{A}$, define $Q(f)$ to be the equivalence class of the roof



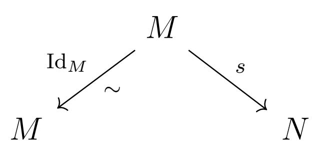
Some verification is required, but this is fact a covariant functor.

Lemma 4.41. *The functor Q above satisfies*

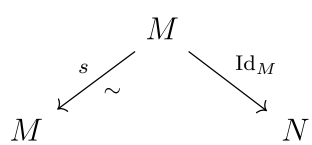
1. *For $s \in S$, $Q(s)$ is an isomorphism in $\mathcal{A}_S$.*
2. *Q is universal with this property. That is, if $F : \mathcal{A} \to \mathcal{B}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$, then there exists a unique functor $G : \mathcal{A}_S \to \mathcal{B}$ such that the following diagram commutes.*



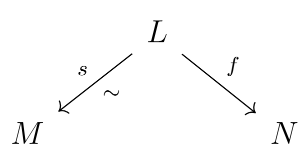
Proof. The first property is easy, since $Q(s)$ is represented by



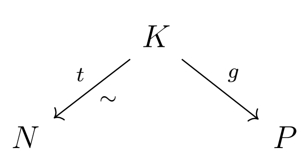
which has inverse represented by



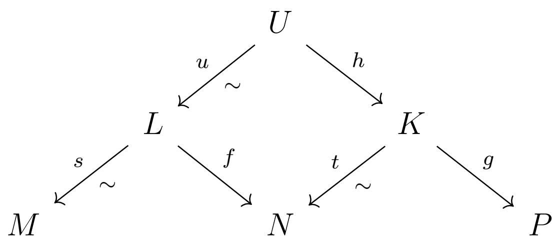
Now for the universal property. Suppose we have a functor $F : \mathcal{A} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for $s \in S$. We define a functor $G : \mathcal{A}_S \to \mathcal{B}$ as follows. On objects, G is the identity. Let $\phi : M \to N$ be a morphism in $\mathcal{A}_S$, and choose a roof representing ϕ.



Define $G(\phi) = F(f) \circ F(s)^{-1}$. It is obvious that $F = G \circ Q$. It is not immediately obvious that G is well defined on equivalence classes of morphisms, but we omit this verification. It is obvious that G takes the identity morphism to the identity morphism, but it remains to check is that G respects composition. Given another morphism $\psi : N \to P$, we represent it also by a roof.



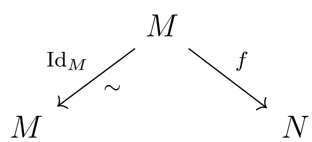
Then the composition $\psi \circ \phi$ is represented by a roof as below, where U, u, h are constructed using (LC3a).



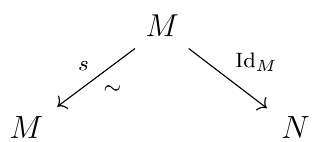
Then assuming that G is well defined with respect to the equivalence of roofs, we can calculate that

$$G(\psi\phi) = F(gh)F(su)^{-1} = F(g)F(h)F(u)^{-1}F(s)^{-1} = F(g)F(t)^{-1}F(f)F(s)^{-1} = G(\psi)G(\phi)$$

Hence G is a functor. Finally, we need to verify uniqueness of G. This is fairly obvious from the diagram which G satisfies, and thinking about what G does to roofs of the form



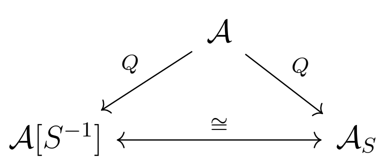
and



□

We summarize much of our preceeding work in the following theorem.

Theorem 4.42. *Let $\mathcal{A}$ be a category and S a localizing class of morphisms in $\mathcal{A}$. Then $Q : \mathcal{A} \to \mathcal{A}_S$ satisfies the same universal property as the localization functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$, so there is a unique isomorphism $\mathcal{A}_S \cong \mathcal{A}[S^{-1}]$ making the following diagram commute.*



Proof. Immediate from lemma 4.41. □

Lemma 4.43. *Let $\mathcal{A}$ be a category and S a localizing class. Two roofs*

are equivalent (as roofs) if and only if the paths

$$M \xleftarrow{s} L \xrightarrow{f} N \qquad M \xleftarrow{t} K \xrightarrow{g} N$$

are equivalent (using elementary transformations).

Proof. No idea how to prove this. ☐

Proposition 4.44. *Two left roofs are equivalent if and only if they represent the same morphism in $\mathcal{A}[S^{-1}]$.*

Proof. If two left roofs are equivalent, lemma 4.35 says that they represent the same morphism, so we just need to prove the converse. Suppose we have two roofs which represent the same morphism $\phi : M \to N$ in $\mathcal{A}_S$.

$$\begin{array}{ccccc} & & L & & \\ & \overset{s}{\swarrow}\sim & & \overset{f}{\searrow} & \\ M & & & & N \end{array} \qquad \begin{array}{ccccc} & & K & & \\ & \overset{t}{\swarrow}\sim & & \overset{g}{\searrow} & \\ M & & & & N \end{array}$$

Under the isomorphism in theorem 4.42, these roofs each respectively correspond to the equivalence classes of the paths

$$M \xleftarrow{s} L \xrightarrow{f} N \qquad M \xleftarrow{t} K \xrightarrow{g} N$$

in $\mathcal{A}[S^{-1}]$. Since they represent the same morphism ϕ, they are equivalent paths, so by lemma 4.43, the roofs are equivalent. ☐

Corollary 4.45. *If S is a localizing class, every morphism in $\mathcal{A}[S^{-1}]$ can be written in the form $Q(f)Q(s)^{-1}$ and in the form $Q(t)^{-1}Q(g)$.*

Proof. Immediate from theorem 4.42. ☐

4.2.4 Localization of subcategories

Let $\mathcal{A}$ be a category and $\mathcal{B}$ a subcategory of $\mathcal{A}$. There is an "embedding" functor

$$\mathcal{B} \to \mathcal{A}$$

which is the identity on both objects and morphisms. This is always faithful, but may fail to be full or essentially surjective. If $\mathcal{B}$ is a full subcategory, then this is full, so it is fully faithful.

Let S be a localizing class in $\mathcal{A}$. We can then consider $S_{\mathcal{B}}$ which is the subclass of S which are morphisms in $\mathcal{B}$. It is not necessarily the case that $S_{\mathcal{B}}$ is a localizing class in $\mathcal{B}$. However, if $S_{\mathcal{B}}$ is a localizing class in $\mathcal{B}$, then the embedding functor extends to a functor

$$\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{A}[S^{-1}]$$

in the only possible reasonable way. It is the identity on objects, and "inclusion" of left roofs. That is to say, a morphism in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ is represented by a left roof, which we can then think of as representing a morphism in $\mathcal{A}[S^{-1}]$.

This induced functor may fail to be faithful. The following proposition gives an additional condition which we can impose in order to make the induced embedding fully faithful.

Proposition 4.46. *Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$ and S a localizing class in $\mathcal{A}$. Suppose that*

1. *$S_{\mathcal{B}}$ is a localizing class in $\mathcal{B}$.*
2. *For any morphism $s : A \to B$ in S with $B \in \mathrm{Ob}(\mathcal{B})$, there exists a morphism $s' : B' \to A$ in $\mathcal{A}$ sith $B' \in \mathrm{Ob}(\mathcal{B})$ such that $ss' \in S_{\mathcal{B}} \subset S$.*[13]

$$B' \xrightarrow{s'} A \xrightarrow{s} B$$

Then the functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{A}[S^{-1}]$ is fully faithful.

Proof. Not very interesting. □

4.3 Localization of additive categories

Next we want to see that localizing an additive category $\mathcal{A}$ results in a localized category $\mathcal{A}[S^{-1}]$ which also has an additive structure. The result is as you would expect - everything works out as well as possible. If $\mathcal{A}$ is additive and S is a localizing class, then we can reuse the same localized category $\mathcal{A}[S^{-1}] \cong \mathcal{A}_S$ and put an additive structure on it, and show that the localization functor Q is an additive functor, and that it is universal among additive functors which take morphisms in S to isomorphisms.

Remark 4.47. If $\mathcal{A}$ is additive, we can restate (LC4'). The new version is: for $f : M \to N$,

$$\exists s \in S, sf = 0 \iff \exists t \in S, tf = 0$$

Our first step towards giving $\mathcal{A}[S^{-1}]$ an additive structure is to define addition of morphisms. Since we think of roofs as "fractions," it makes sense that in order to "add" roofs we would need to find "common denominators." The next lemma takes care of this. It doesn't even require additivity of the category $\mathcal{A}$.

Lemma 4.48. *Let $\mathcal{A}$ be a category and S a localizaing class of morphisms in $\mathcal{A}$. Suppose we have a collection of roofs*

$$\begin{array}{ccc} & L_i & \\ {\scriptstyle s_i}\swarrow{\scriptstyle \sim} & & \searrow{\scriptstyle f_i} \\ M & & N \end{array}$$

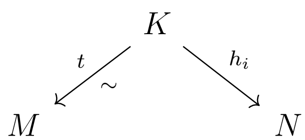
representing morphisms $\phi_i : M \to N$ in $\mathcal{A}[S^{-1}]$, for $i = 1, 2, \ldots, n$. Then there exists an object L in $\mathcal{A}$, a morphism $s : L \to M$ in S, and morphisms $g_i : L_i \to N$ such that the roof

$$\begin{array}{ccc} & L & \\ {\scriptstyle s}\swarrow{\scriptstyle \sim} & & \searrow{\scriptstyle g_i} \\ M & & N \end{array}$$

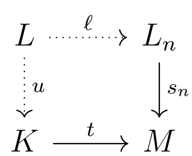
[13] Since $\mathcal{B}$ is a full subcategory, requiring $ss' \in S_{\mathcal{B}}$ is equivalent to requiring $ss' \in S$.

represents the morphism ϕ_i.

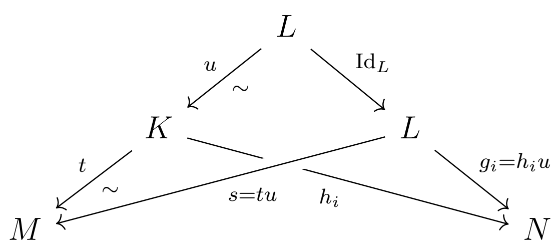
Proof. We induct on n. The case $n = 1$ is vacuous, so assume $n \geq 2$. We are given n roofs involving L_i, s_i, f_i representing morphisms ϕ_i for $i = 1, \ldots, n$. By inductive hypothesis, for $i = 1, \ldots, n-1$, we have an object K, a morphism $t : K \to M$ in S, and morphisms $h_i : K \to N$ so that ϕ_i is represented by the roof



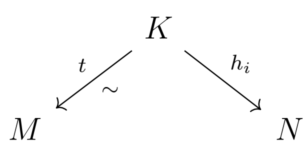
By (LC3a), there exists an object L and morphisms $u : L \to K, \ell : L \to L_n$ with $u \in S$, making the following diagram commute.

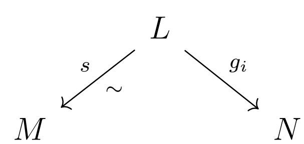


Set $s = tu = s_n\ell : L \to M$, and note this is in S. For $i = 1, \ldots, n-1$ set $g_i = h_i u$ and set $g_n = f_n\ell$. Then for $i = 1, \ldots, n-1$ the diagram

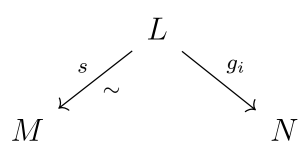


gives an equivalence between the roofs

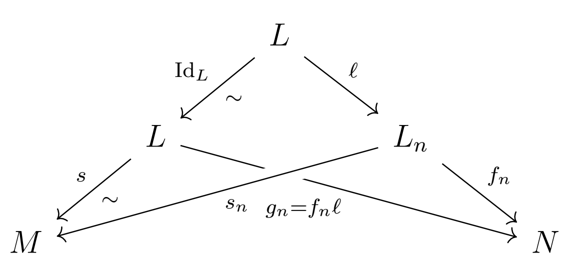




so for $i = 1, \ldots, n-1$, the roof



represents the morphism ϕ_i. For $i = n$, the diagram



gives an equivalence between the roofs

$$M \xleftarrow[\sim]{s} L \xrightarrow{g_n = f_n \ell} N \qquad M \xleftarrow[\sim]{s_n} L \xrightarrow{f_n} N$$

Since the roof on the right represents ϕ_n, so does the roof on the left. This completes the induction. □

The preceeding result allows us to define addition of roofs if $\mathcal{A}$ is additive.

Definition 4.49. Let $\mathcal{A}$ be an additive category and S a localizing class in $\mathcal{A}$. Let $\phi, \psi : M \to N$ be morphisms in $\mathcal{A}[S^{-1}]$. By the preceeding lemma, we can represent ϕ, ψ using roofs of the form

$$M \xleftarrow[\sim]{s} L \xrightarrow{f} N \qquad M \xleftarrow[\sim]{s} L \xrightarrow{g} N$$

We then define $\phi + \psi$ to be the equivalence class of the roof

$$M \xleftarrow[\sim]{s} L \xrightarrow{f+g} N$$

where $f + g$ is the addition in $\mathrm{Hom}_{\mathcal{A}}(L, N)$. It is not trivial to check that this is well defined on equivalence classes, but we omit the details. As long as it is well defined, it is obvious that this gives an abelian group structure to $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ since all the properties (identity, associativity, commutativity, inverses) are inherited from $\mathrm{Hom}_{\mathcal{A}}(M, N)$.

Proposition 4.50. *Let $\mathcal{A}$ be an additive category and S a localizing class of morphisms in $\mathcal{A}$. Then $\mathcal{A}[S^{-1}]$ is an additive category. Specifically,*

1. *For any objects M, N, P, the composition operation*

$$\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N) \times \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(N, P) \to \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, P)$$

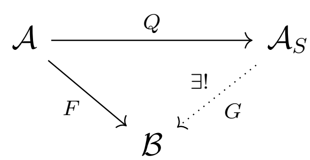
   is additive in each variable.

2. *The zero object in $\mathcal{A}[S^{-1}]$ is the same zero object as in $\mathcal{A}$.*

3. *The biproduct of two objects M, N in $\mathcal{A}[S^{-1}]$ is the same biproduct $M \oplus N$ from $\mathcal{A}$, with slightly modified canonical morphisms.*

4. *The localization functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is additive.*

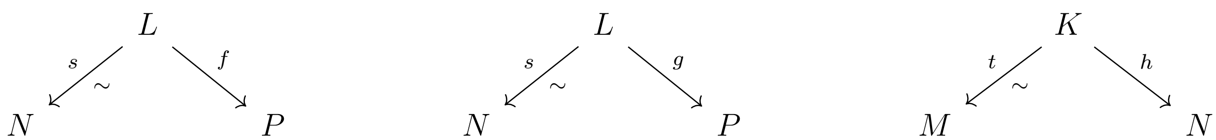
5. *The localization functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is universal among additive functors which take S to isomorphisms. That is, if $F : \mathcal{A} \to \mathcal{B}$ is an additive functor such that $F(s)$ is an isomorphism for every $s \in S$, then there exists a unique additive functor $G : \mathcal{A}_S \to \mathcal{B}$ such that the following diagram commutes.*[14]



Proof. (1) First we address additivity in the left variable. Let $\phi, \psi : N \to P$ and $\chi : M \to N$ be morphisms in $\mathcal{A}[S^{-1}]$. We need to show that

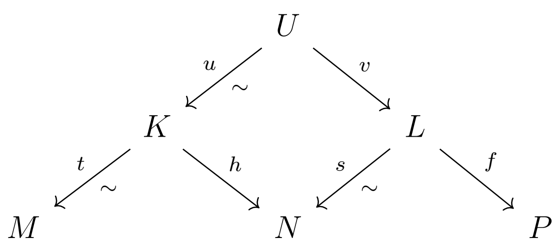
$$(\phi + \psi)\chi = \phi\chi + \psi\chi$$

Using lemma 4.48, we can represent ϕ, ψ by roots with a common denominator s.

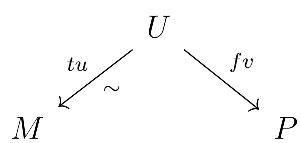


ϕ ψ χ

Then by (LC3a), there exist morphisms $u : U \to K$ and $v : U \to L$ so that $sv = hu$ and $u \in S$.



Thus the composition $\phi\chi$ is represented by the roof



$\phi\chi$

Similarly, $\psi\chi$ is represented by the roof

[14] This universal property is very subtly different than the universal property we already know that Q satisfies. The only differences are that now we assume F is additive, and that the constructed functor G is additive.

$$M \xleftarrow[\sim]{tu} U \xrightarrow{gv} P$$

$$\psi\chi$$

Then by definition of addition of roofs, $\phi\chi + \psi\chi$ is represented by the roof

$$M \xleftarrow[\sim]{tu} U \xrightarrow{fv+gv} P$$

$$\phi\chi + \psi\chi$$

We know that $\phi + \chi$ is represented by the roof

$$N \xleftarrow[\sim]{s} L \xrightarrow{f+g} P$$

$$\phi + \chi$$

so $(\phi + \psi)\chi$ is represented by

$$M \xleftarrow[\sim]{tu} U \xrightarrow{(f+g)v} P$$

$$(\phi + \psi)\chi$$

Since $\mathcal{A}$ is additive, $(f + g)v = fv + gv$, so $(\phi + \psi)\chi$ and $\phi\chi + \psi\chi$ are represented by the same roof, hence are equal morphisms in $\mathcal{A}[S^{-1}]$. This proves that composition in $\mathcal{A}[S^{-1}]$ is additive in the left argument. A similar argument works for additivity in the right argument. It is a bit more complicated because the addition is happening on the left side of the roofs, in the "denominator" of the "fraction," but it's not too much more complicated, so we omit it.

(2) Omitted, not very complicated.

(3) Omitted, not very interesting.

(4) Obvious.

(5) Mostly obvious. Just examine the construction of the functor G in lemma 4.41, and additivity of G follows from the fact that F is additive. □

4.3.1 Kernel of the localization functor

We continue to discuss the situation where $\mathcal{A}$ is an additive category, S is a localizaing class of morphisms in $\mathcal{A}$, and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is the localization functor. We want to describe the "kernel" of the localization functor Q, that is, which morphisms and objects in $\mathcal{A}$ are sent to zero morphisms and the zero object of $\mathcal{A}[S^{-1}]$.

First we recall what happens with rings. In a commutative associative unital ring R with a multiplicative subset S, the localization morphism $\epsilon : R \to R_S$ has kernel

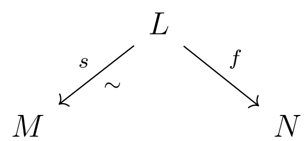
$$\ker \epsilon = \{a \in R : sa = 0 \text{ for some } s \in S\}$$

In the noncommutative case, where R is only an associative unital ring, the localization still exists but we have far less control over and knowledge about the localization R_S and the localization morphism ϵ. However, we did show that if S is a right denominator set, then the kernel still has the same description as above, though the description is not symmetric in the sense that $sa = 0$ and $as = 0$ are not equivalent. If S is both a left and right denominator set, then

$$\ker \epsilon = \{a \in R : sa = 0 \text{ for some } s \in S\} = \{a \in R : as = 0 \text{ for some } s \in S\}$$

so in this situation we regain some symmetry in the description of the kernel. Since we modeled the definition of localizing class on right and left denominator sets, it should not be too surprising that we get an analogous description of when morphisms in the localized category $\mathcal{A}[S^{-1}]$ are zero.

Lemma 4.51. *Let $\mathcal{A}$ be an additive category and S a localizing class in $\mathcal{A}$. Let $\phi : M \to N$ be a morphism in $\mathcal{A}[S^{-1}]$, represented by a left roof*

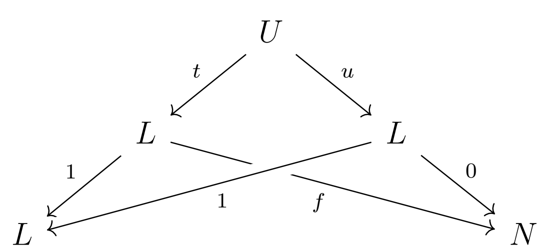


The following are equivalent.

1. $\phi = 0$ *(the zero morphism in* $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$*)*
2. *There exists $t \in S$ such that $ft = 0$.*
3. *There exists $t \in S$ such that $tf = 0$.*

Proof. The equivalence (2) $\iff$ (3) is immediate from (LC4) or (LC4').

(1) $\implies$ (2) Suppose $\phi = 0$. From the description of ϕ as a left roof, $\phi = Q(f)Q(s)^{-1} = 0$. Since $Q(s)$ is an isomorphism, this implies $Q(f) = 0$. That is to say, we have an equivalence of roofs



Since this is an equivalence of roofs, $1 \circ t = t \in S$, and by commutativity of the diagram $ft = 0$.

(2) $\implies$ (1) Suppose $ft = 0$ for some $t \in S$. Then $Q(f)Q(t) = 0$. Since $Q(t)$ is an isomorphism, $Q(f) = 0$. So $\phi = Q(f)Q(s)^{-1} = 0$. □

Corollary 4.52. *Let $f : M \to N$ be a morphism in $\mathcal{A}$. The following are equivalent.*

1. $Q(f) = 0$
2. *There exists $t \in S$ such that $ft = 0$.*
3. *There exists $t \in S$ such that $tf = 0$.*

Proof. Apply previous lemma with $\phi = Q(f)$. □

The previous corollary addresses the question of when a morphism becomes zero in the localized category, now we consider when objects become isomorphic to the zero object.

Corollary 4.53. *Let M be an object in $\mathcal{A}$. The following are equivalent.*

1. *$Q(M)$ is (isomorphic to) the zero object (in $\mathcal{A}[S^{-1}]$).*
2. *There exists an object N in $\mathcal{A}$ such that the zero morphism $N \xrightarrow{0} M$ is in S.*
3. *There exists an object N in $\mathcal{A}$ such that the zero morphism $M \xrightarrow{0} N$ is in S.*

Proof. (1) $\implies$ (2) Suppose $Q(M) \cong 0$. Then $Q(\mathrm{Id}_M) = 0$. By the previous corollary, this implies that there exists a morphism $t : N \to M$ with $t \in S$ so that $t \circ \mathrm{Id}_M = 0$, but this just says $t = 0$.

(1) $\implies$ (3) Similar to (1) $\implies$ (2).

(2) $\implies$ (1) Suppose $N \xrightarrow{0} M$ is an isomorphism. Then the composition

$$Q(M) \xrightarrow{Q(0)} Q(N) \xrightarrow{Q(0)^{-1}} Q(M)$$

is both the zero morphism and the identity on $Q(M)$, so $Q(M)$ must be the zero object.

(3) $\implies$ (1) Similar to (2) $\implies$ (1). □

4.4 Localization of abelian categories

Next up we discuss localization in the situation where $\mathcal{A}$ is abelian. We'll eventually prove

Theorem 4.54. *Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. Then $\mathcal{A}[S^{-1}]$ is abelian, and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is exact.* [15]

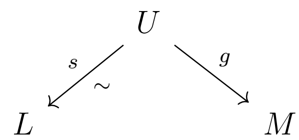
Before this, some build up of lemmas. First, a lemma regarding how the localization functor Q acts on epi- and monomorphisms.

Proposition 4.55. *Let $\mathcal{A}$ be an additive category and S a localizing class in $\mathcal{A}$, and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ the localization functor. Let $f : M \to N$ be a morphism in $\mathcal{A}$.*

1. *If f is a monomorphism, then $Q(f)$ is a monomorphism.*
2. *If f is an epimorphism, then $Q(f)$ is an epimorphism.*

[15] Probably there is an analogous universal property in which the functor G is exact, but I'm not sure.

Proof. We prove (1), the proof of (2) is similar, or can be obtained by passing to the opposite category. Suppose $\phi : L \to M$ is a morphism in $\mathcal{A}[S^{-1}]$ such that $Q(f)\phi = 0$. To show that $Q(f)$ is a monomoprhism, it suffices to show that $\phi = 0$. Represent ϕ by a left roof



so $\phi = Q(g)Q(s)^{-1}$. Then

$$0 = Q(f)\phi = Q(f)Q(g)Q(s)^{-1} = Q(fg)Q(s)^{-1}$$

Since $Q(s)$ is an isomorphism, this implies $Q(fg) = 0$. Then by corollary 4.52, there exists $t \in S$ so that $fgt = 0$ (this is an equality of morphisms in $\mathcal{A}$). Since f is a monomorphism, $gt = 0$. So

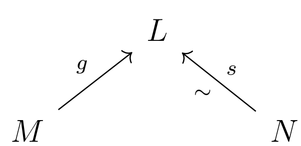
$$0 = Q(gt) = Q(g)Q(t)$$

Since $Q(t)$ is an isomorphism, this implies $Q(g) = 0$. So $\phi = Q(g)Q(s)^{-1} = 0$. $\square$

Lemma 4.56. *Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. Every morphism in $\mathcal{A}[S^{-1}]$ has a kernel and cokernel.*

Before the proof, we note for future reference that the proof says that basically the kernel of a morphism is the kernel of the "numerator." That is, a morphism ϕ in $\mathcal{A}[S^{-1}]$ can be written as $Q(s)^{-1}Q(g)$ with $s \in S$, and the kernel of ϕ is essentially (the image of) the kernel of g (under Q).

Proof. Let $\phi : M \to N$ be a morphism in $\mathcal{A}[S^{-1}]$, and right it as a right roof, so $\phi = Q(s)^{-1}Q(g)$ with $s \in S$.

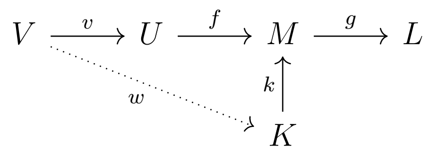


Since $Q(s)$ is an isomorphism, a morphism $\chi : K \to M$ is the kernel of ϕ if and only if it is the kernel of $Q(g)$. Since $\mathcal{A}$ is abelian, $g : M \to L$ has a kernel in $\mathcal{A}$, which we denote by $k : K \to M$. We define $\chi = Q(k)$, and claim that χ is the kernel of $Q(g)$ in $\mathcal{A}[S^{-1}]$.

To show that χ is the kernel, we need to show that any $\psi : P \to M$ (in $\mathcal{A}[S^{-1}]$) such that $Q(g)\psi = 0$ factors through $Q(K)$. Let $\psi : P \to M$ be a morphism such that $Q(g)\psi = 0$. We can write ψ as a left roof, $\psi = Q(f)Q(t)^{-1}$ with $t \in S$. Then

$$0 = Q(g)\psi = Q(g)Q(f)Q(t)^{-1} = Q(gf)Q(t)^{-1}$$

Since $Q(t)$ is an isomorphism, we get $Q(gf) = 0$. Then by corollary 4.52, there exists $v : V \to U$ with $v \in S$ so that $gfv = 0$. Thus fv factors through the kernel of g. That is, there exists a (unique) morphism $w : V \to K$ making the following diagram commute.



Then applying Q to this,

$$Q(kw) = Q(fv) \implies Q(k)Q(w) = Q(f)Q(v) \implies Q(f) = Q(k)Q(w)Q(v)^{-1} = \chi Q(w)Q(v)^{-1}$$

Then we can write ψ as

$$\psi = Q(f)Q(t)^{-1} = \chi Q(w)Q(t)Q(v)^{-1}$$

so ψ factors through χ. Finally, we just need to verify that the factorization is unique. Suppose $\psi = \chi\alpha = \chi\beta$ for morphisms α, β. Then $\chi(\alpha - \beta) = 0$. Since $k : K \to M$ is a monomorphism, by proposition 4.55, χ is a monomorphism so this implies $\alpha - \beta = 0$, or $\alpha = \beta$. □

Remark 4.57. The previous proof says a bit more than that morphisms in $\mathcal{A}[S^{-1}]$ have kernels and cokernels. It says $Q(\ker f) \cong \ker Q(f)$, and similarly Q takes cokernels to cokernels. This happens for both the objects and morphisms associated with the kernel/cokernel. Because of the universal property, this isomorphism isn't just any isomorphism, it's a unique isomorphism, so we can really identified $Q(\ker f)$ with $\ker Q(f)$.

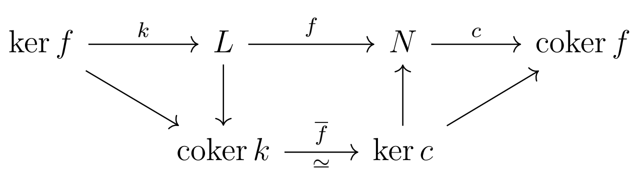
Recall that in an additive category $\mathcal{A}$ with all kernels and cokernels, every morphism with $\phi : M \to N$ induces a morphism

$$\overline{\phi} : \operatorname{coker} \ker \phi \to \ker \operatorname{coker} \phi$$

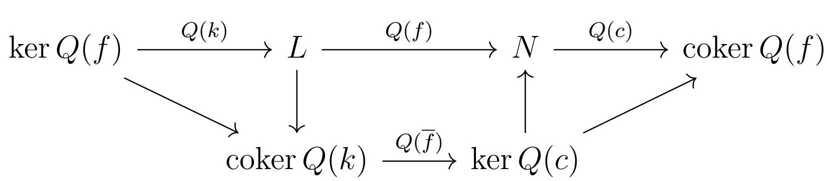
and that the category $\mathcal{A}$ is abelian if and only if ϕ is strict, meaning that $\overline{\phi}$ is an isomorphism.

Lemma 4.58. *Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. Every morphism in $\mathcal{A}[S^{-1}]$ is strict.*

Proof. Let $\phi : M \to N$ be a morphism in $\mathcal{A}[S^{-1}]$ and represent it by a left roof, so $\phi = Q(f)Q(s)^{-1}$ with $s \in S$, for some morphisms $f : L \to N$ and $s : L \to M$. Since $\mathcal{A}$ is abelian, f is strict. We depict this in the following commutative diagram.



We can apply Q to this diagram. By remark 4.57, we can identify $Q(\ker f)$ with $\ker Q(f)$ (and identify the associated morphisms). On objects $Q(L) = L$ and $Q(N) = N$ so we omit the Q's for those.



Because of all these natural identifications, we also identify $Q(\overline{f}) = \overline{Q(f)}$. Then

$$\overline{\phi} = \overline{Q(f) \circ Q(s)^{-1}} = \overline{Q(f)} \circ \overline{Q(s)^{-1}}$$

must be an isomorphism, because $Q(s)^{-1}, \overline{Q(s)^{-1}}$ are isomorphisms. □

Corollary 4.59. *If $\mathcal{A}$ is abelian and S is a localizing class in $\mathcal{A}$, then $\mathcal{A}[S^{-1}]$ is abelian.*

Finally, we want to show that the localization functor Q is exact.

Lemma 4.60. *Let $\mathcal{A}$ be an abelian category and S a localizing class in $\mathcal{A}$. The localization functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ is exact.*

Proof. Let $M \xrightarrow{f} N \xrightarrow{g} P$ be an exact sequence in $\mathcal{A}$, so $gf = 0$ and the natural map $\operatorname{im} f = \operatorname{coker} \ker f = \ker \operatorname{coker} f \to N$ is $\ker g$. When we apply Q to obtain

$$QM \xrightarrow{Qf} QN \xrightarrow{Qg} QP$$

it is clear that $Qg \circ Qf = 0$. Since we also know Q preserves kernels and cokernels, $Q(\operatorname{im} f) \to QN$ is the the kernel of Qg. Thus Q is exact. □

Remark 4.61. The preceeding lemma provides the last step in the proof of Theorem 4.54.

4.4.1 Thick subcategories

Definition 4.62. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a full subcategory. $\mathcal{B}$ is a **thick** subcategory if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, we have

$$Y \in \operatorname{ob}(\mathcal{B}) \iff X, Z \in \operatorname{ob}(\mathcal{B})$$

Equivalently, $\mathcal{B}$ is closed under taking subobjects, quotients, and extensions.

Remark 4.63. A thick subcategory must contain the zero object, and be closed under isomorphisms using the short exact sequences

$$0 \to N \xrightarrow{\cong} M \to 0 \to 0$$

Sometimes a full subcategory which is closed under isomorphisms of objects is called a **strictly full** subcategory.

Remark 4.64. Some sources refer to our definition of a thick subcategory above as a **Serre subcategory**, but also sometimes Serre subcategory means something slightly different than this. We'll stick to the terminology "thick."

Lemma 4.65. *A thick subcategory of an abelian category is abelian.*

Proof. Mostly obvious. Clearly the thick subcategory contains all kernels and cokernels because it contains all subobjects and quotients. It contains all biproducts because the biproduct is an extension. Since every morphism in the ambient category is strict, this also applies to the thick subcategory. □

Definition 4.66. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a subcategory. We define $\boldsymbol{S_{\mathcal{B}}}$ to be the class of morphisms $M \xrightarrow{f} N$ in $\mathcal{A}$ such that the objects $\ker f, \operatorname{coker} f$ are in $\mathcal{B}$.

The next lemma describes a categorical analog of the "kernel" of the localization functor on objects. It says that the "object kernel" of Q is a thick subcategory, and that conversely, any thick subcategory is the "object kernel" of a suitable localization functor.

Lemma 4.67. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a full subcategory.*

1. *Suppose S is a localizing class in $\mathcal{A}$, with localization functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$. If $QM \cong 0$ for every $M \in \operatorname{ob}(\mathcal{B})$, then $\mathcal{B}$ is thick.*

2. *Suppose $\mathcal{B}$ is thick. Then $S_{\mathcal{B}}$ is a localizing class in $\mathcal{A}$.*

3. *Suppose $\mathcal{B}$ is thick and let $Q : \mathcal{A} \to \mathcal{A}[S_{\mathcal{B}}^{-1}]$ be the localization functor. Let $M \in \operatorname{ob}(\mathcal{A})$. Then $M \in \operatorname{ob}(\mathcal{B})$ if and only if $QM \cong 0$.*

Proof. (1) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathcal{A}$. Since Q is exact, $0 \to QX \to QY \to QZ \to 0$ is exact in $\mathcal{A}[S^{-1}]$. So $QY \cong 0$ if and only if if $QX \cong 0$ and $QZ \cong 0$.

(2) For any identity morphism in $\mathcal{A}$, the kernel and cokernel are zero which lie in $\mathcal{B}$, so $S_{\mathcal{B}}$ satisfies (LC1). Next we show (LC2). Let $s : L \to M$ and $t : M \to N$ are in $S_{\mathcal{B}}$. Consider the pullback square

$$\begin{array}{ccc} P & \xrightarrow{g} & \ker t \\ \downarrow{\scriptstyle f} & & \downarrow \\ N & \xrightarrow{s} & M \end{array}$$

which all exists in $\mathcal{A}$ since abelian categories always have pullbacks. Because the square is cartesian, it follows that $P \cong \ker(ts)$. Also, f identifies $\ker g$ and $\ker s$, so $\ker g$ is an object of $\mathcal{B}$. Since $\operatorname{coker} \ker g$ is a subobject of $\ker t$, it is also in $\mathcal{B}$. Now consider the short exact sequence

$$0 \to \ker g \to P \to \operatorname{coker} \ker g \to 0$$

The outer objects lie in $\mathcal{B}$, so the middle one does as well because $\mathcal{B}$ is thick. Then it follows that $ts \in S_{\mathcal{B}}$. This completes the proof of (LC2). The arguments for (LC3a) and (LC3b) are really technical and boring, so we skip them. The proofs mostly involve a lot of thinking carefully about more pullback squares.

(3) Suppose $M \in \operatorname{ob}(\mathcal{B})$. Then $M \xrightarrow{s} 0$ has kernel M and the zero object as cokernel, so $s \in S_{\mathcal{B}}$. Then $Q(s)$ is an isomorphism $QM \cong 0$ in $\mathcal{A}[S_{\mathcal{B}}^{-1}]$. Conversely, suppose $QM \cong 0$. Then there exists an object N in $\mathcal{A}$ such that $N \xrightarrow{0} M$ is in $S_{\mathcal{B}}$. So the cokernel of this morphisms which is M, is in $\mathcal{B}$. □

Example 4.68. Let $\mathcal{A}$ be the abelian category of vector spaces over a field F, and let $\mathcal{B}$ be the full subcategory of finite dimensional vector spaces. It is not hard to see that $\mathcal{B}$ is thick. The localizing class $S_{\mathcal{B}}$ consists of all linear maps $f : V \to W$ such that $\ker f$ and $\operatorname{coker} f = W/f(V)$ are finite dimensional. Informally, $S_{\mathcal{B}}$ consists of all linear maps that are "almost" injective and "almost" surjective, in the sense that they are injective and surjective up to finite dimensional subspaces.

Definition 4.69. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a thick subcategory. The localized category $\mathcal{A}[S_{\mathcal{B}}^{-1}]$ is called the **quotient category** and denoted $\mathcal{A}/\mathcal{B}$.

Theorem 4.70 (Gabriel 1962)**.** *Let $\mathcal{B}$ be a thick subcategory of an abelian category $\mathcal{A}$. Then there exists an abelian category $\mathcal{A}/\mathcal{B}$ and an exact functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ with the following universal property: For any abelian category $\mathcal{C}$ and exact functor $F : \mathcal{A} \to \mathcal{C}$ such that $FM \cong 0$ for all $M \in \text{ob}(\mathcal{B})$, there exists a unique exact functor $G : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ such that $F = GQ$.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad Q \quad} & \mathcal{A}/\mathcal{B} \\ & {\scriptstyle F}\searrow & \swarrow{\scriptstyle \exists!\, G} \\ & \mathcal{C} & \end{array}$$

Remark 4.71. The way Gabriel originally constructed the category $\mathcal{A}/\mathcal{B}$ is different from our localization methods, but we give a proof using localizations.

Proof. Let F be as in the statement of the theorem. We show that F takes $S_{\mathcal{B}}$ to isomorphisms in $\mathcal{A}[S_{\mathcal{B}}^{-1}] = \mathcal{A}/\mathcal{B}$, so it satisfies the hypothesis of proposition 4.50 part (5). Suppose $M \xrightarrow{f} N$ is a morphism in $\mathcal{A}$ in $S_{\mathcal{B}}$. Then f can be factored as

$$M \xrightarrow{g} L \xrightarrow{h} N$$

with g an epimorphism and h a monomorphism. Then consider the short exact sequences

$$0 \to \ker f = \ker g \to M \xrightarrow{g} L \to 0$$
$$0 \to L \xrightarrow{h} N \to \operatorname{coker} f = \operatorname{coker} h \to 0$$

Since F is exact, we can apply it to these and get exact sequences

$$0 \to F(\ker f) \to FM \xrightarrow{Fg} FL \to 0$$
$$0 \to FL \xrightarrow{Fh} FN \to F(\operatorname{coker} f) \to 0$$

Since $f \in S_{\mathcal{B}}$, we know that $\ker f, \operatorname{coker} f \in \text{ob}(\mathcal{B})$. Then they become isomorphic to zero in $\mathcal{A}[S_{\mathcal{B}}^{-1}]$ by lemma 4.67 part (3). Thus Fg and Fh are isomorphisms, so $Ff = F(gh) = (Fh)(Fg)$ is an isomorphism. Then using the universal property from proposition 4.50, there exists a unique additive functor $G : \mathcal{A}/\mathcal{B} = \mathcal{A}[S_{\mathcal{B}}^{-1}] \to \mathcal{C}$ making the required diagram in Gabriel's theorem commute. All that remains to verify is that G is exact, but we omit this part of the proof. □

4.5 Localization of triangulated categories

Our next goal is to show that if we start with a triangulated category and a localizing class of morphisms, then the localization also has a triangulated structure, provided we impose two additional assumptions on our localizing class. Roughly speaking, these will be

1. The localizing class is invariant under the translation functor.

2. A jazzed up version of (TR3) involving morphisms in our localizing class.

Definition 4.72. Let $\mathcal{C}$ be a triangulated category with translation functor $T_{\mathcal{C}} = [1]$, and let S be a localizing class in $\mathcal{C}$. S is **compatible with triangulation** if it satisfies

(LT1) For a morphism s in $\mathcal{C}$, $s \in S \iff s[1] \in S$.

(LT2) Any diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle t} & & & & \big\downarrow{\scriptstyle s[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

with distinguished rows and $s, t \in S$ can be completed to a morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle s[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

The axiom (LT1) says that S is invariant under the translation functor and its inverse.

Definition 4.73. Let $\mathcal{C}$ be a triangulated category and S a localizing class which is compatible with triangulation, and let $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the localization functor. Consider the functor

$$F := QT_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}[S^{-1}] \qquad M \mapsto Q(M[1])$$

By (LT1), for $s \in S$, we know $T_{\mathcal{C}}(s) \in S$. So $F(s) = Q(T_{\mathcal{C}}(s))$ is an isomorphism for every $s \in S$. Thus by the universal property of the localization, there exists a unique functor $G : \mathcal{C}[S^{-1}] \to \mathcal{C}[S^{-1}]$ making the following diagram commute.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad Q \quad} & \mathcal{C}[S^{-1}] \\ & {\scriptstyle F} \searrow \quad \swarrow {\scriptstyle G} & \\ & \mathcal{C}[S^{-1}] & \end{array}$$

We denote G by $T_{\mathcal{C}[S^{-1}]}$. Using the definition of F, we expand the previous diagram.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T_{\mathcal{C}}} & \mathcal{C} \\ \big\downarrow{\scriptstyle Q} & & \big\downarrow{\scriptstyle Q} \\ \mathcal{C}[S^{-1}] & \xrightarrow{T_{\mathcal{C}[S^{-1}]}} & \mathcal{C}[S^{-1}] \end{array}$$

So we can characterize $T_{\mathcal{C}[S^{-1}]}$ as the unique functor making the diagram above commute. Similarly, we can consider the composition of functors $QT_{\mathcal{C}}^{-1} : \mathcal{C} \to \mathcal{C}[S^{-1}]$, and apply the same universal property to find a functor $\mathcal{C}[S^{-1}] \to \mathcal{C}[S^{-1}]$ which is inverse to $T_{\mathcal{C}[S^{-1}]}$. So $T_{\mathcal{C}[S^{-1}]}$ is an automorphism of $\mathcal{C}[S^{-1}]$. We call it the **induced translation functor**.

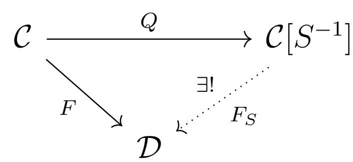
Definition 4.74. Let $\mathcal{C}, S$ be as above. We define a triangle in $\mathcal{C}[S^{-1}]$ to be distinguished if it is isomorphic to the image of a distinguished triangle from $\mathcal{C}$ under the localization functor Q. In other words, after applying Q to any distinguished triangle in $\mathcal{C}$, we obtain a distinguished triangle in $\mathcal{C}[S^{-1}]$, and any triangle isomorphic to such a triangle is also considered distinguished.

Theorem 4.75. *Let $\mathcal{C}$ be a triangulated category and S a localizing class compatible with triangulation. Then $\mathcal{C}[S^{-1}]$ with the translation functor and distinguished triangles as defined above is triangulated.*

Proof. (TR1a), (TR1b), (TR2) are all fairly immediate from the triangulated properties of $\mathcal{C}$. (TR1c), (TR3), and (TR4) are all somewhat involved technical proofs, so we skip the details. □

Remark 4.76. It is immediate from the construction that the localization functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ commutes with translation (see definition 3.9) and takes distinguished triangles to distinguished triangles, so Q is an exact functor (see definition 3.10).

Theorem 4.77. *Let $\mathcal{C}, \mathcal{D}$ be triangulated categories and $F : \mathcal{C} \to \mathcal{D}$ an exact functor. Let S be a localizing class in $\mathcal{C}$ which is compatible with triangulation such that $F(s)$ is an isomorphism (in $\mathcal{D}$) for every $s \in S$. Then there exists a unique exact functor $F_S : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = F_S Q$.*



Proof. By theorem 4.50, we know there exists a unique additive functor F_S making the required diagram commute, so it suffices to show that the functor constructed in that theorem is exact.

We give only a brief sketch. First, one shows that F_S commutes with translation, mostly just using the fact that F commutes with translation, but this is slightly tricky. The fact that F_S takes distinguished triangles in $\mathcal{C}[S^{-1}]$ to distinguished triangles in $\mathcal{D}$ is mostly immediate from the construction, since by definition every distinguished triangle in $\mathcal{C}[S^{-1}]$ comes from one in $\mathcal{C}$ (up to isomorphism), and F takes that distinguished triangle in $\mathcal{C}$ to one in $\mathcal{D}$. □

Recall that a functor $F : \mathcal{C} \to \mathcal{A}$ from a triangulated category to an abelian category is **cohomological** if it takes distinguished triangles in $\mathcal{C}$ to exact sequences in $\mathcal{A}$. More precisely, if

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle in $\mathcal{C}$, then

$$FX \to FY \to FZ$$

is exact in $\mathcal{A}$. The next proposition says that cohomological functors on a triangulated category always induce cohomological functors on the localized category.

Proposition 4.78. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ an abelian category, and let S be a localizing class in $\mathcal{C}$ which is compatible with triangulation. Let $F : \mathcal{C} \to \mathcal{A}$ be a cohomological functor such that $F(s)$ is an isomorphism for all $s \in S$. Then there exists a unique cohomological functor $F_S : \mathcal{C}[S^{-1}] \to \mathcal{A}$ such that $F = F_S Q$.*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad Q \quad} & \mathcal{C}[S^{-1}] \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \exists!\, F_S} & \\ & \mathcal{D} & \end{array}$$

Proof. From theorem 4.50, we already know that a unique additive functor F_S exists. We just need to show that F_S is cohomological. Let $X \to Y \to Z \to X[1]$ be a distinguished triangle in $\mathcal{C}[S^{-1}]$. Then there exists a distinguished triangule $U \to V \to W \to U[1]$ in $\mathcal{C}$ and an isomorphism of distinguished triangles in $\mathcal{C}[S^{-1}]$ as below.

$$\begin{array}{ccccccc} QU & \longrightarrow & QV & \longrightarrow & QW & \longrightarrow & QU[1] \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Then we apply F_S to this diagram to obtain the following commutative diagram in $\mathcal{A}$.

$$\begin{array}{ccccc} F_S QU & \longrightarrow & F_S QV & \longrightarrow & F_S QW \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\ F_S X & \longrightarrow & F_S Y & \longrightarrow & F_S Z \end{array}$$

The top row of this is just $FU \to FV \to FW$, which is exact because F is cohomological. So the bottom row is also exact, hence F_S is cohomological. $\square$

5 Derived categories

Finally we get our goal in all of this, the derived category. We fix an abelian category $\mathcal{A}$, with chain complex category $C(\mathcal{A})$ and homotopy category $K(\mathcal{A})$.

Remark 5.1. Recall that if two chain maps are chain homotopic, they induce the same map on cohomology. So if f is a quasi-isomorphism and f is homotopic to g, then g is also a quasi-isomorphism. Hence it makes sense to talk about quasi-isomorphisms in $K(\mathcal{A})$, since being a quasi-isomorphism is independent of the homotopy class representative.

Definition 5.2. Let S be the class of quasi-isomorphisms in $K(\mathcal{A})$. The **derived category** of $\mathcal{A}$, denoted $D(\mathcal{A})$, is the localization $K(\mathcal{A})[S]$.

Because of theorem 4.15, this localization exists and comes with a localization functor Q with a suitable universal property. However, we can't do much with the localization unless we know that S is more than just some class of morphisms. In particular, we would like to know that S is a localizing class, and compatible with the triangulation on $K(\mathcal{A})$. Before we get to that, a short lemma regarding quasi-isomorphisms.

Lemma 5.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a distinguished triangle in $K(\mathcal{A})$.*

1. *f is a quasi-isomorphism if and only if Z is acyclic.*
2. *g is a quasi-isomorphism if and only if X is acyclic.*

Proof. (1) f is a quasi-isomorphism if and only if the cone C_f is acyclic, and any distinguished triangle in $K(\mathcal{A})$ is isomorphism to a cone triangle. In particular Z is isomorphic to C_f, so Z is acyclic if and only if C_f is acyclic. (2) Similar argument as (1), or just apply rotation to (1). □

Theorem 5.4. *The class S of quasi-isomorphisms in $K(\mathcal{A})$ is a localizing class compatible with triangulation.*

Proof. Throughout this proof, we denote objects and morphisms in $K(\mathcal{A})$ without the usual dot to indicate grading, since it would mostly just clutter the notation.

(LC1) Obvious, the identity morphism is a quasi-isomorphism.

(LC2) Composition of quasi-isomorphisms is obviously a quasi-isomorphism.

(LC3a) Suppose we have a diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{s} & Y \end{array}$$

in $K(\mathcal{A})$, with s a quasi-isomorphism. To verify (LC3a) we need to complete this to a diagram

$$\begin{array}{ccc} X & \overset{t}{\dashrightarrow} & Z \\ \vdots{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{s} & Y \end{array}$$

with t being a quasi-isomorphism. First, we use (TR1c) to extend s to a distinguished triangle.

$$X \xrightarrow{s} Y \xrightarrow{i} U \xrightarrow{p} X[1]$$

We know U is isomorphic to the cone of s, but this is not so important. Anyway, since s is a quasi-isomorphism, U is acyclic by lemma 5.3. We apply (TR2) to rotate an obtain a distinguished triangle

$$Y \xrightarrow{i} U \xrightarrow{p} X[1] \xrightarrow{-s[1]} Y[1]$$

Now consider the composition $if = i \circ f : Z \to U$. Again using (TR1c) we extend this to a distinguished triangle.

$$Z \xrightarrow{if} U \xrightarrow{r} V \xrightarrow{u} Z[1]$$

Now we consider the following diagram, with the left square commuting.

$$\begin{array}{ccccccc}
Z & \xrightarrow{if} & U & \xrightarrow{r} & V & \xrightarrow{u} & Z[1] \\
\big\downarrow f & & \big\downarrow \mathrm{Id} & & & & \big\downarrow f[1] \\
Y & \xrightarrow{i} & U & \xrightarrow{p} & X[1] & \xrightarrow{-s[1]} & Y[1]
\end{array}$$

By (TR3), we can complete this to a morphism of triangles, so there is a morphism $v : V \to X[1]$ making the following diagram commute.

$$\begin{array}{ccccccc}
Z & \xrightarrow{if} & U & \xrightarrow{r} & V & \xrightarrow{u} & Z[1] \\
\big\downarrow f & & \big\downarrow \mathrm{Id} & & \big\downarrow v & & \big\downarrow f[1] \\
Y & \xrightarrow{i} & U & \xrightarrow{p} & X[1] & \xrightarrow{-s[1]} & Y[1]
\end{array}$$

Now apply (TR2) again to rotate this backwards, and obtain

$$\begin{array}{ccccccc}
V[-1] & \xrightarrow{-u[1]} & Z & \xrightarrow{if} & U & \xrightarrow{v} & V \\
v[-1] \big\downarrow & & \big\downarrow f & & \big\downarrow \mathrm{Id} & & \big\downarrow v \\
X & \xrightarrow{s} & Y & \xrightarrow{i} & U & \xrightarrow{p} & X[1]
\end{array}$$

So we take $W = V[-1]$ and $t = -u[1]$ and $g = v[-1]$ and obtain the needed completed diagram needed for (LC3a). The morphism $t = -u[1]$ is a quasi-isomorphism because U is acyclic.

$$\begin{array}{ccc}
V[-1] & \xrightarrow{-u[1]} & Z \\
v[-1] \big\downarrow & & \big\downarrow f \\
X & \xrightarrow{s} & Y
\end{array}$$

(LC3b) Analogous argument to (LC3a).

(LC4) Since $K(\mathcal{A})$ is additive, we can work with (LC4') instead. Suppose we have a morphism $X \xrightarrow{f} Y$ in $K(\mathcal{A})$ and a quasi-isomorphism $s : Y \to Z$ such that $sf = 0$. We need to construct a quasi-isomorphism $t : W \to X$ such that $ft = 0$. Extend s to a distinguished triangle $Y \xrightarrow{s} Z \to U \to Y[1]$, and consider the folllowing diagram with distinguished rows.

$$
\begin{array}{ccccccc}
X & \longrightarrow & 0 & \longrightarrow & X[1] & \xrightarrow{-\operatorname{Id}_{X[1]}} & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{s} & Z & \xrightarrow{i} & U & \xrightarrow{p} & Y[1]
\end{array}
$$

By (TR3) we can complete this to a morphism of triangles. We denote the new morphism by $-v$.

$$
\begin{array}{ccccccc}
X & \longrightarrow & 0 & \longrightarrow & X[1] & \xrightarrow{-\operatorname{Id}_{X[1]}} & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow & & \vdots{\scriptstyle -v} & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{s} & Z & \xrightarrow{i} & U & \xrightarrow{p} & Y[1]
\end{array}
$$

Now we apply the inverse translation functor to the right square and obtain

$$
\begin{array}{ccc}
X & \xrightarrow{-\operatorname{Id}_X} & X \\
{\scriptstyle -v[-1]}\downarrow & & \downarrow{\scriptstyle f} \\
U[-1] & \xrightarrow{p[-1]} & Y
\end{array}
$$

So $f = p[-1] \circ v[-1]$. Now extend $v[-1]$ to a distinguished triangle,

$$X \xrightarrow{v[-1]} U[-1] \to V \xrightarrow{t} X[1]$$

Then rotate this triangle backwards to obtain a new distinguished triangle

$$V[-1] \xrightarrow{-t[-1]} X \xrightarrow{v[-1]} U[-1] \to V$$

We know that $v[-1] \circ t[-1] = 0$ because this triangle is distinguished. Also, U is acyclic, so $U[-1]$ is acyclic, so t is a quasi-isomorphism. Thus

$$f \circ t[-1] = p[-1] \circ v[-1] \circ t - [-1] = 0$$

This proves one direction of (LC4'). The other direction is analogous, so we skip it.

(LT1) It's obvious that the translation of a quasi-isomorphism is still a quasi-isomorphism.

(LT2) Suppose we have a diagram as below with distinguished rows.

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}
$$

with s, t quasi-isomorphisms. We know that from (TR3) we can complete this to a morphism of triangles with morphism $u : Z \to Z'$.

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \vdots{\scriptstyle u} & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}
$$

To verify (LT2), we need to prove that u is a quasi-isomorphism. Both rows being distinguished gives rise to long exact sequences on cohomology, and these long exact sequences have induced morphisms from s, t, u making the following diagram commute.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^n X & \longrightarrow & H^n Y & \longrightarrow & H^n Z & \longrightarrow & H^{n+1} X & \longrightarrow & H^{n+1} Y & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle H^n s} & & \big\downarrow{\scriptstyle H^n t} & & \big\downarrow{\scriptstyle H^n u} & & \big\downarrow{\scriptstyle H^{n+1} s} & & \big\downarrow{\scriptstyle H^{n+1} t} & & \\
\cdots & \longrightarrow & H^n X' & \longrightarrow & H^n Y' & \longrightarrow & H^n Z' & \longrightarrow & H^{n+1} X' & \longrightarrow & H^{n+1} Y' & \longrightarrow & \cdots
\end{array}$$

In the diagram above, the rows are exact, and the morhisms induced by s, t are isomorphism because s, t are quasi-isomorphisms. So we can apply the 5-lemma, which says that $H^n u$ is an isomorphism. This works for every $n \in \mathbb{Z}$, so u is a quasi-isomorphism. □

Remark 5.5. To summarize, we have finished showing that the class S of quasi-isomorphism in the homotopy category $K(\mathcal{A})$ forms a localizing class compatible with triangulation. This has several useful implications for the derived category $D(\mathcal{A}) := K(\mathcal{A})[S^{-1}]$, such as

- $D(\mathcal{A})$ is triangulated
- The localization functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ is exact and has the universal property of theorem 4.77
- The localization functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ has the universal property of proposition 4.78. In particular, the cohomology functor $H^n : K(\mathcal{A}) \to \mathcal{A}$ is cohomological, so it factors through Q and through the derived category, which is to say, there are cohomological functors $H^n : D(\mathcal{A}) \to \mathcal{A}$ which agree with $H^n : K(\mathcal{A}) \to \mathcal{A}$ on the image of Q.

Definition 5.6. The derived category $D(\mathcal{A})$ has all the same associated bounded subcategories analogous to those in $K(\mathcal{A})$. More precisely, we have categories

$$D^*(\mathcal{A}) := K^*(\mathcal{A})[(S^*)^{-1}]$$

for $* \in \{+, -, b\}$, which are respectively bounded above, bounded below, and bounded full subcategories.

For the next theorem, recall the functor

$$\mathcal{A} \to C(\mathcal{A}) \qquad X \mapsto \cdots \to 0 \to X \to 0 \to \cdots$$

which takes an object to a complex concentrated in degree zero. This also induces $\mathcal{A} \to K(\mathcal{A})$ and $D : \mathcal{A} \to D(\mathcal{A})$. When the image is $C(\mathcal{A})$, the functor is obviously fully faithful, but when it lands in $K(\mathcal{A})$, this is no longer true. But when we go further to $D(\mathcal{A})$, we recover this fully faithfulness.

Theorem 5.7. *The functor*

$$D : \mathcal{A} \to D(\mathcal{A}) \qquad X \mapsto \cdots \to 0 \to X \to 0 \to \cdots$$

is fully faithful.

Proof. We need to show that for objects M, N of $\mathcal{A}$, the map induced by the functor D on hom-sets is bijective. (These are additive categories, so this is a morphism of abelian groups.)

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{D(\mathcal{A})}(DM, DN)$$

We start with injectivity. Let $\phi, \psi : M \to N$ be morphisms in $\mathcal{A}$ such that $D\phi = D\psi$ as morphisms $DM \to DN$. By remark 5.5, the functor $H^0 : K(\mathcal{A}) \to \mathcal{A}$ induces $H^0 : D(\mathcal{A}) \to \mathcal{A}$.

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{\quad Q \quad} & D(\mathcal{A}) \\ & {\scriptstyle H^0}\searrow \quad \swarrow{\scriptstyle H^0} & \\ & \mathcal{A} & \end{array}$$

For any morphism $\phi : M \to N$, we have $H^0(D\phi) = \phi$, so $\phi = H^0(D\phi) = H^0(D\psi) = \psi$, thus D is injective on morphisms.

Now for surjectivity. Suppose $\phi : DM \to DN$ is a morphism in $D(\mathcal{A})$. Represent ϕ by a left roof.

$$\begin{array}{ccc} & X & \\ {\scriptstyle s}\swarrow{\scriptstyle \sim} & & \searrow{\scriptstyle f} \\ DM & & DN \end{array}$$

where s is a quasi-isomorphism. So $\phi = Q(f)Q(s)^{-1}$, with s, f morphisms in $K(\mathcal{A})$. Since DM is concentrated in degree zero and s is a quasi-isomorphism, $H^nX = 0$ for all $n \neq 0$. Write the complex X as

$$X \qquad \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots$$

and let Y be the complex

$$Y \qquad \cdots X^{-1} \xrightarrow{d^{-1}} \ker d^0 \to 0 \to \cdots$$

Then we have a natural morphism $r : Y \to X$ depicted below which is a quasi-isomorphism, where the morphism $\ker d^0 \to X^0$ is the natural map associated with the kernel.

$$\begin{array}{cccccccccc} Y & & \cdots & \longrightarrow & X^{-1} & \longrightarrow & \ker d^0 & \longrightarrow & 0 & \longrightarrow \cdots \\ \downarrow{\scriptstyle r} & & & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & \\ X & & \cdots & \longrightarrow & X^{-1} & \xrightarrow{d^{-1}} & X^0 & \xrightarrow{d^0} & X^1 & \longrightarrow \cdots \end{array}$$

Since r is a quasi-isomorphism, it is invertible in $D(\mathcal{A})$. Let $s' = sr$ and $f' = fr$. So the left roof which we represented ϕ by is equivalent to the left roof

$$\begin{array}{ccc} & Y & \\ {\scriptstyle s'=sr}\swarrow{\scriptstyle \sim} & & \searrow{\scriptstyle f'=fr} \\ DM & & DN \end{array}$$

where Y is a complex satisfies $Y^n = 0$ for $n > 0$. Now let $R = H^0X \in \text{ob}(\mathcal{A})$. We can write s' as a composition $s' = ut$ where t is a quasi-isomorphism and u is an isomorphism of complexes.

$$\begin{array}{ccccccccccc}
Y & & \cdots & \longrightarrow & X^{-1} & \xrightarrow{d^{-1}} & \ker d^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow t & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR & & \cdots & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow u & & & & \downarrow & & \downarrow & & \downarrow & & \\
DM & & \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \qquad (s' : Y \to DM)$$

Similarly, we can factor f' as $f' = gt$ where t is a quasi-isomorphism.

$$\begin{array}{ccccccccccc}
Y & & \cdots & \longrightarrow & X^{-1} & \xrightarrow{d^{-1}} & \ker d^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow t & & & & \downarrow & & \downarrow & & \downarrow & & \\
DR & & \cdots & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow g & & & & \downarrow & & \downarrow & & \downarrow & & \\
DN & & \cdots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array} \qquad (f' : Y \to DN)$$

So the roof

$$DM \xleftarrow[\sim]{s'=ut} Y \xrightarrow{f'=gt} DN$$

is equivalent to the roof

$$DM \xleftarrow[\sim]{u} DR \xrightarrow{g} DN$$

Finally, since u is an isomorphism of complexes, we have an equivalence of roofs

$$\begin{array}{ccccc}
 & & DR & & \\
 & \overset{1}{\swarrow}\sim & & \overset{u}{\searrow} & \\
 & DR & & DR & \\
\overset{u}{\swarrow}\sim & & \overset{1}{\swarrow}\ \ \overset{g}{\searrow} & & \overset{gu^{-1}}{\searrow} \\
DM & & & & DN
\end{array}$$

Putting everything together, our original map ϕ is represented by the roof

$$DM \xleftarrow[\sim]{1} DR \xrightarrow{gu^{-1}} DN$$

which is precisely $D(gu^{-1})$. Thus $\phi = D(gu^{-1})$, so D is also surjective on morphisms. $\square$

Corollary 5.8. *Let $\mathcal{A}$ be an abelian category. If $D(\mathcal{A})$ is abelian, then $\mathcal{A}$ is semisimple.*

Proof. Suppose $D(\mathcal{A})$ is abelian. Since it is also triangulated, it is semisimple. To show that $\mathcal{A}$ is semisimple, we show that every epimorphism in $\mathcal{A}$ splits.

Let $\alpha : M \to N$ be an epimorphism. Then α induces an epimorphism between complexes concentrated in degree zero in $K(\mathcal{A})$. Since the localization functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ is exact the image $D(\alpha)$ is then an epimorphism. Since $D(\mathcal{A})$ is semisimple, $D(\alpha)$ splits, i.e. there is a section $\beta : DN \to DM$ such that $\beta \circ D(\alpha) = \mathrm{Id}_{DM}$.

Since $\mathcal{A} \to D(\mathcal{A})$ is full, $\widetilde{\beta}$ comes from a morphism in $\mathcal{A}$. That is, there is a morphism $\widetilde{\beta} : N \to M$ such that $D(\widetilde{\beta}) = \beta$. Then

$$D(\widetilde{\beta} \circ \alpha) = D(\widetilde{\beta}) \circ D(\alpha) = \beta \circ D(\alpha) = \mathrm{Id}_{DM} = D(\mathrm{Id}_M)$$

Since D is faithful, this implies $\widetilde{\beta} \circ \alpha = \mathrm{Id}_M$, which is to say α splits. Hence $\mathcal{A}$ is semisimple. □

Lemma 5.9. *Let $\mathcal{A}$ be an abelian category, and let X, Y be objects in $C(\mathcal{A})$. If there exists $n \in \mathbb{Z}$ such that*

$$X^p = 0, \;\; p \geq n \qquad \text{and} \qquad Y^p = 0, \;\; \forall p < n$$

then $\mathrm{Hom}_{D(\mathcal{A})}(X, Y) = 0$.

Remark 5.10. Before proving this, note that under the hypotheses, it is obvious that $\mathrm{Hom}_{C(\mathcal{A})}(X, Y) = 0$ since any morphism $f^p : X^p \to Y^p$ has at least one of domain or codomain being zero. Since $\mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ is a quotient of this, it is also immediately zero. On the other hand, since $D(\mathcal{A})$ is a localization of $K(\mathcal{A})$, and morphisms in the localization are constructed as paths or roofs, it's not a priori impossible that $\mathrm{Hom}_{D(\mathcal{A})}(X, Y)$ could have some nonzero morphisms. Of course, the lemma says that it does not in this case.

Proof. Let $\phi : X \to Y$ be a morphism in $D(\mathcal{A})$, and represent ϕ by a left roof.

$$\begin{array}{ccccc} & & Z & & \\ & \overset{s}{\underset{\sim}{\swarrow}} & & \overset{f}{\searrow} & \\ X & & & & Y \end{array}$$

where s is a quasi-isomorphism. We know that $H^p(X) = 0$ for $p \geq n$, so s being a quasi-isomorphism means $H^p(Z) = 0$ for $p \geq n$ as well. Let U be the following truncation of Z.

$$\cdots \to Z^{n-2} \xrightarrow{d^{n-2}} \ker d^{n-1} \to 0 \to \cdots$$

Then the natural map $i : U \to Z$ is a quasi-isomorphism, so the roof for ϕ is equivalent to

$$\begin{array}{ccccc} & & U & & \\ & \overset{si}{\underset{\sim}{\swarrow}} & & \overset{fi}{\searrow} & \\ X & & & & Y \end{array}$$

But then we have a morphism $fi : U \to Y$ which is a morphism in $K(\mathcal{A})$, and $U^p = 0$ for $p \geq n$ and $Y^p = 0$ for $p < n$, so $fi = 0$. Hence $\phi = 0$. □

Proposition 5.11. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be a short exact sequence in $C(\mathcal{A})$. Then there exists a morphism $h : Z \to X[1]$ in $D(\mathcal{A})$ such that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle in $D(\mathcal{A})$. Moreoever, if the original sequence is all concentrated in degree zero, then h is unique (in fact $h = 0$ in this scenario).

Proof. Let C_f be the cone of f, so $C_f = X[1] \oplus Y$. Set $m : C_f \to Z$ to be the composition

$$C_f \xrightarrow{\text{proj}} Y \xrightarrow{g} Z$$

where proj is the projection map. We claim that

1. m is a morphism of complexes.
2. Let $i_f : Y \to C_f$ be the natural morphism. Then $mi_f = g$.
3. m is a quasi-isomorphism.

(1) Note that it is not immediate that m is a morphism of complexes, because $C_f \xrightarrow{\text{proj}} Y$ is not, in general, a morphism of complexes. To show that m is a morphism of complexes, we need to verify that $m^{n+}d^n_{C_f} = d^n_Z m^n$. We use our matrix notational shortcut.

$$m^{n+1}d^n_{C_f} = \begin{pmatrix} 0 & g^{n+1} \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = \begin{pmatrix} g^{n+1}f^{n+1} & g^{n+1}d_Y^n \end{pmatrix} = \begin{pmatrix} 0 & d_Z^n g^n \end{pmatrix} = d_Z^n m$$

The key step here is that exactness of the origianl sequence means that the left component $g^{n+1}f^{n+1}$ vanishes.

(2) This is immediate from the definition of m.

(3) Consider the trivial commutative diagram of complexes

$$\begin{array}{ccc} X & \xrightarrow{\text{Id}} & X \\ \downarrow{\scriptstyle \text{Id}} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

This induces a morphism between the cones of the horizontal maps, so we have $w : C_{\text{Id}_X} \to C_f$, which is given in degree n by

$$w^n = \begin{pmatrix} \text{Id}_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix}$$

Since f is a monomorphism, so is w. For any $n \in \mathbb{Z}$,

$$\operatorname{im} w^n = X^{n+1} \oplus \operatorname{im} f^n \cong X^{n+1} \oplus \ker g^n = \ker m^n$$

so we have a short exact sequence of complexes

$$0 \to C_{\mathrm{Id}_X} \xrightarrow{w} C_f \xrightarrow{m} Z \to 0$$

Since Id_X is a quasi-isomorphism, C_{Id_X} is acyclic. So from looking at the associated long exact sequence in cohomology associated to the above seqeuence of complexes, it follows that m is a quasi-isomorphism.

This completes our discussion of the properties of m, so we can now complete the proof of the proposition. Since m is a quasi-isomorphism, is it is an isomorphism in $D(\mathcal{A})$. We also have the projection $p_f : C_f \to X[1]$ which is a morphism of complexes. Set $h = p_f m^{-1} : Z \to X[1]$ in $D(\mathcal{A})$. This fits into the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i_f} & C_f & \xrightarrow{p_f} & X[1] \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{Id}} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

Since the top row is a distinguished triangle and the vertical arrows are isomorphisms in $D(\mathcal{A})$, the bottom is as well.

Finally, we tackle the uniqueness statement. Suppose that our short exact sequence $0 \to X \to Y \to Z \to 0$ of complexes is concentrated in degree zero, which is to say, it comes from a short exact sequence $0 \to X^0 \to Y^0 \to Z^0 \to 0$ in A. We have our morphism $h : Z \to X[1]$, but Z and $X[1]$ don't have any nonzero terms in the same degrees, so by lemma 5.9 $\mathrm{Hom}_{D(\mathcal{A})}(Z, X[1]) = 0$. Thus there is a unique morphism $Z \to X[1]$ (the zero morphism), which must be h. □

Remark 5.12. The morphism h from the previous proposition is not unique, in general. We only know uniqueness in the somewhat restrictive situation mentioned in the proposition.

5.1 Truncation

Definition 5.13. Recall that if X is a chain complex and $n \in \mathbb{Z}$, the truncations of X are defined by

$$\tau_{\leq n}(X)^m = \begin{cases} X^m & m < n \\ \ker d^m & m = n \\ 0 & m > n \end{cases}$$

and

$$\tau_{\geq n}(X)^m = \begin{cases} 0 & m < n \\ \operatorname{coker} d^{m-1} & m = n \\ X^m & m > n \end{cases}$$

We discussed previously how the truncation functors $\tau_{\leq n}, \tau_{\geq n} : C(\mathcal{A}) \to C(\mathcal{A})$ behave well with respect to chain homotopies, so they induce truncation functors $\tau_{\leq n}, \tau_{\geq n} : K(\mathcal{A}) \to K(\mathcal{A})$.

We would like these to also pass to the derived category, and this works out nicely. If $X \xrightarrow{s} Y$ is a quasi-isomorphism, then the truncated versions $\tau_{\leq n}(s)$ and $\tau_{\geq n}(s)$ are also quasi-isomorphisms, so by the universal property we obtain truncation functors $\tau_{\leq n}, \tau_{\geq n} : D(\mathcal{A}) \to D(\mathcal{A})$.

Remark 5.14. Let X be a chain complex in $\mathcal{A}$. Define a complex Q by

$$Q^m = \begin{cases} 0 & m < n \\ \operatorname{coim} d^m & m = n \\ X^m & m > n \end{cases}$$

where $\operatorname{coim} d^m = \operatorname{coker} \ker d^m$ is the coimage, and boundary maps in Q are induced by those in X. Then we have a short exact sequence of complexes

$$0 \to \tau_{\leq n}(X) \to X \to Q \to 0$$

We also have the following commutative diagram, which describes a quasi-isomorphism $Q \to \tau_{\geq n+1}(X)$.

$$\begin{array}{ccccccccccccc}
Q & & \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{coim} d^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \cong} & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
\tau_{\geq n+1}(X) & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} d^n & \longrightarrow & X^{n+2} & \longrightarrow & \cdots
\end{array}$$

From our short exact sequence of complexes above, using proposition 5.11 we obtain a distinguished triangle

$$\tau_{\leq n}(X) \to X \to Q \to \tau_{\leq n}(X)[1]$$

But as we said before, in $D(\mathcal{A})$, $Q \cong \tau_{\geq n+1}(X)$, so we can write this as

$$\tau_{\leq n}(X) \to X \to \tau_{\geq n+1}(X) \xrightarrow{h} \tau_{\leq n}(X)[1]$$

But from lemma 5.9 we know that $\operatorname{Hom}_{D(\mathcal{A})}(\tau_{\leq n}(X), \tau_{\geq n+1}(X)[-1]) = 0$, so the map h from proposition 5.11 is unique (and is in fact zero).

The preceding remark serves as a proof for the following proposition.

Proposition 5.15. *For $X \in \operatorname{ob}(C(\mathcal{A}))$ and $n \in \mathbb{Z}$, there exists a unique morphism*[16]

$$h : \tau_{\geq n+1}(X) \to \tau_{\leq n}(X)[1]$$

in $D(\mathcal{A})$ such that the following is a distinguished triangle.

$$\tau_{\leq n}(X) \to X \to \tau_{\geq n+1}(X) \xrightarrow{h} \tau_{\leq n}(X)[1]$$

Remark 5.16. My professor didn't say anything about h being zero in the previous proposition, so maybe I've made a mistake. It seems like a much less interesting result if h is just zero.

[16] I'm pretty sure h just has to be the zero morphism, actually.

Definition 5.17. Recall that for $* \in \{+, -, b\}$ we have the categories $C^*(\mathcal{A})$ of bounded below, bounded above, and bounded chain complexes respectively. To each is a corresponding bounded homotopy category $K^*(\mathcal{A})$. In $K^*(\mathcal{A})$, we have a localizing class S^* of quasi-isomorphims, which is compatible with triangulation, so we have an associated bounded derived category

$$D^*(\mathcal{A}) = K^*(\mathcal{A})[(S^*)^{-1}]$$

There are also "inclusion" functors

$$K^+(\mathcal{A}) \to K(\mathcal{A}) \qquad K^-(\mathcal{A}) \to K(\mathcal{A})$$

which by the universal property of localization induce analogous functors

$$D^+(\mathcal{A}) \to D(\mathcal{A}) \qquad D^-(\mathcal{A}) \to D(\mathcal{A})$$

Our next objective is to establish that these functors are fully faithful and injective (up to isomorphism) on objects. First, we state a lemma without proof.

Lemma 5.18. *Let $\mathcal{C}$ be an additive category. Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, and let S be a localizing class in $\mathcal{C}$. Assume that*

1. *$S_\mathcal{B} = S \cap \mathrm{Mor}(\mathcal{B})$ is a localizing class in $\mathcal{B}$.*
2. *For any morphism $N \xrightarrow{s} M$ in S with $M \in \mathrm{ob}(\mathcal{B})$, there exists a morphism $P \xrightarrow{n} N$ in $\mathcal{C}$ with $P \in \mathrm{ob}(\mathcal{B})$ such that $sn \in S$.*

Then the natural functor $\mathcal{B}[S_\mathcal{B}^{-1}] \to \mathcal{C}[S^{-1}]$ is fully faithful.

Proof. This isn't terribly complicated, just work with left roofs. □

Remark 5.19. The conclusion of the previous lemma also holds replacing condition (2) with the "dual" condition

(2') For any morphism $M \xrightarrow{s} N$ with $s \in S$ and $M \in \mathrm{ob}(\mathcal{B})$, there exists a morphism $N \xrightarrow{n} P$ in $\mathcal{C}$ with $P \in \mathrm{ob}(\mathcal{B})$ such that $ns \in S$.

Proposition 5.20. *The inclusion functors $D^+(\mathcal{A}) \to D(\mathcal{A}), D^-(\mathcal{A}) \to D(\mathcal{A})$ are fully faithful and injective on objects.*

Proof. We'll just do the argument for D^-, the argument for D^+ is analogous. We want to apply lemma 5.18 in the situation $\mathcal{C} = K(\mathcal{A}), \mathcal{B} = K^-(\mathcal{A})$. Then $\mathcal{B}[S_\mathcal{B}^{-1}] = K^-(\mathcal{A})[(S^+)^{-1}] = D^-(\mathcal{A})$, so if the hypotheses of the lemma hold, then $D^-(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful. We already know that condition (1) holds in this situation, so it just remains to verify condition (2).

Suppose $N \xrightarrow{s} M$ is a quasi-isomorphism with $M \in \mathrm{ob}(K^-(\mathcal{A}))$. Then there exists $n \in \mathbb{Z}$ such that $H^p(M) = 0$ for all $p > n$. Since s is a quasi-isomorphism, $H^p(N) = 0$ for $p > n$ as well. Thus the natural morphism $i : \tau_{\leq n}(N) \to N$ is a quasi-isomorphism. So $si : \tau_{\leq n}(N) \to M$ is a quasi-isomorphism. Since $\tau_{\leq n}(N) \in \mathrm{ob}(K^-(\mathcal{A}))$, this verifies property (2).

Injectivity on objects is obvious. If X, Y are complexes in $D^-(\mathcal{A})$ which become isomorphic in $D(\mathcal{A})$, that same isomorphism is a morphism between X, Y in $D^-(\mathcal{A})$. □

Remark 5.21. In light of the previous proposition, we view $D^+(\mathcal{A}), D^-(\mathcal{A})$ as full subcategories of $D(\mathcal{A})$. Also note that we have functors $D^b(\mathcal{A}) \to D^-(\mathcal{A})$ and $D^b(\mathcal{A}) \to D^+(\mathcal{A})$ and by the same arguments these are fully faithful and injective on objects. In fact, $D^b(\mathcal{A}) = D^+(\mathcal{A}) \cap D^-(\mathcal{A})$ just as in the chain complex or homotopy categories.

5.2 Injective resolutions

For most of this section, we assume that $\mathcal{A}$ is an abelian category with enough injectives.

Remark 5.22. Let X be an object of $\mathcal{A}$, and take an injective resolution of X.

$$X \to I^0 \to I^1 \to I^2 \to \cdots$$

We can think of this as a quasi-isomoprhism in $D^+(\mathcal{A})$.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

Theorem 5.25 below is a direct generalization of this.

Remark 5.23. Recall the general construction of pushouts in an abelian category. If we have a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \\ C & & \end{array}$$

define $\theta = (\alpha, -\gamma) : A \to B \oplus C$ and $D = \operatorname{coker} \theta = B \sqcup_A C$. Then there are natural maps $\beta : B \to D, \delta : C \to D$ (associated with the coproduct $B \oplus C$) making the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \vdots{\scriptstyle\beta} \\ C & \overset{\delta}{\cdots\rightarrow} & D \end{array}$$

If we assume $\mathcal{A}$ is a category of R-modules (using Freyd-Mitchell) then we have the following useful property of pushout diagrams: given $b \in B, c \in C$ such that $\delta(c) = \beta(b)$, there exists $a \in A$ such that $b = \alpha(a), c = \gamma(a)$.

Proposition 5.24. *Let $\mathcal{A}$ be an abelian category*[17]*, and suppose $\mathcal{B}$ is a class of objects in $\mathcal{A}$ containing the zero object, and such that every object of $\mathcal{A}$ admits a monomorphism into an object of $\mathcal{B}$. Then given $X \in \operatorname{ob}(C(\mathcal{A}))$ such that $X^n = 0$ for $n < 0$, there exist a complex Y with $Y^n \in \mathcal{B}$ for all n, $Y^n = 0$ for $n < 0$, and a quasi-isomorphism $X \xrightarrow{s} Y$.*

[17] It doesn't matter if $\mathcal{A}$ has enough injectives for this result.

Proof. Using the Freyd-Mitchell embedding, we may assume $\mathcal{A}$ is a category of R-modules, to simplify the proof. Write X as

$$0 \to X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots$$

Let $s^0 : X^0 \to Y^0$ be a monomorphism with $Y^0 \in \mathcal{B}$. Now take the pushout of s^0, d_X^0.

$$\begin{array}{ccc} X^0 & \xrightarrow{d_X^0} & X^1 \\ \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle \alpha} \\ Y^0 & \xrightarrow{u} & Y^0 \sqcup_{X^0} X^1 \end{array}$$

Now let $i_1 : Y^0 \sqcup_{X^0} X^1 \to Y^1$ be a monomorphism with $Y^1 \in \mathcal{B}$, and set $d_Y^0 = i^1 u$ and $s^1 = i^1 \alpha$. Then we have a commutative diagram

$$\begin{array}{ccc} X^0 & \xrightarrow{d_X^0} & X^1 \\ \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^1} \\ Y^0 & \xrightarrow{d_Y^0} & Y^1 \end{array}$$

So s^0 induces a morphism $\ker d_X^0 \to \ker d_Y^0$, which is an isomorphism between zero-th homology of the complexes $0 \to X^0 \to X^1$ and $0 \to Y^0 \to Y^1$. So at this point we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^1} & & & & \\ 0 & \longrightarrow & Y^0 & \xrightarrow{d_Y^0} & Y^1 & & & & \end{array}$$

with $Y^i \in \mathcal{B}$ and s^0 inducing an isomorphism on homology. Now we argue inductively. Suppose we have constructed Y^n and s^n fitting into a diagram below, with s^i inducing isomorphisms on homology for $i < n$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^0 & \longrightarrow & \cdots & \longrightarrow & X^n & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle s^0} & & & & \downarrow{\scriptstyle s^n} & & \\ 0 & \longrightarrow & Y^0 & \longrightarrow & \cdots & \longrightarrow & Y^n & & \end{array}$$

We need to construct Y^{n+1}, s^{n+1}, such that the diagram commutes, and s^n is an isomorphism on homology. First, form the pushout $\operatorname{coker} d_Y^{n-1} \sqcup_{X^n} X^{n+1}$, then use the hypothesis of the proposition to obtain a monomorphism from this pushout to an object $Y^{n+1} \in \mathcal{B}$. We illustrate this in the following diagram.

$$\begin{array}{ccccccc} & & X^n & \xrightarrow{d_X^n} & X^{n+1} & & \\ & {\scriptstyle s^n}\swarrow & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ Y^n & \longrightarrow & \operatorname{coker} d_Y^{n-1} & \xrightarrow{\delta} & \operatorname{coker} d_Y^{n-1} \sqcup_{X^n} X^{n+1} & \longrightarrow & Y^{n+1} \end{array}$$

Now define $d_Y^n : Y^n \to Y^{n+1}$ as the composition of the three arrows along the bottom row of the above diagram. Since the composition $Y^{n-1} \to Y^n \to \operatorname{coker} d_Y^{n-1}$ is zero, $d_Y^n d_Y^{n-1} = 0$, so this inductive definition makes Y into a complex. We also need to verify that s^n induces an isomorphism $H^n(X) \to H^n(Y)$.

First, we prove that the induced map on H^n is surjective. Let $y \in \ker \delta \subset \operatorname{coker} d_Y^{n-1}$. So both y and 0 in X^{n+1} are mapped to 0 in the pushout, so there exists $x \in X^n$ such that $d_X^n(x) = 0$ and $\alpha(x) = y$. Since α is the induced map on H^n, this shows $H^n(s)$ is surjectivity.

Finally, we verify that the induced map on H^n is injective. Suppose $x \in X^n$ such that $\alpha(x) = 0$. Then $s^n(x) = d_Y^{n-1}(y)$ for some $y \in Y^{n-1}$. Then consider the pushout at the previous step $n-1$.

$$
\begin{array}{ccccccc}
 & & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & & \\
 & \swarrow^{s^{n-1}} & \downarrow \alpha' & & \vdots\,\beta' & & \\
Y^{n-1} & \longrightarrow & \operatorname{coker} d_Y^{n-2} & \xrightarrow{\delta'} & \operatorname{coker} d_Y^{n-2} \sqcup_{X^{n-1}} X^n & \dashrightarrow & Y^n
\end{array}
$$

We have $x \in X^n$, so $\beta'(x) \in \operatorname{im} \delta'$, which says that $\beta'(x) = \delta'(z)$ for some $z \in \operatorname{coker} d_Y^{n-2}$. Then there exists $\widetilde{x} \in X^{n-1}$ such that $d_X^{n-1}(\widetilde{x}) = 0$. But then $x = 0$ in $H^n(X)$. This proves injectivity. $\square$

Theorem 5.25. *Let $\mathcal{A}$ be an abelian category with enough injectives. Given a complex $X \in \operatorname{ob}(C^+(\mathcal{A}))$, let $i \in \mathbb{Z}$ such that $X^n = 0$ for $n < i$. Then there exists a complex I of injective objects such that $I^n = 0$ for $n < i$ and a quasi-isomorphism $X \xrightarrow{s} I$.*

Proof. Immediate from previous proposition, taking $\mathcal{B}$ to be the class of injective objects and translating X so that $i = 0$. $\square$

Definition 5.26. For a fixed abelian category $\mathcal{A}$, let $\mathcal{I}$ be the full subcategory of $\mathcal{A}$ of injective objects. Since finite coproducts of injective objects are injective, this is is an additive subcategory.

We can then consider the full additive subcategory $K^+(\mathcal{I})$ of $K^+(\mathcal{A})$. This category $K^+(\mathcal{I})$ is invariant under translation, and also for any morphism in $K^+(\mathcal{I})$, the cone of that morphism lies in $K^+(\mathcal{I})$.

Definition 5.27. Let $\mathcal{C}$ be a triangulated category with translation functor $T_{\mathcal{C}}$. A fully additive subcategory $\mathcal{D}$ of $\mathcal{C}$ is **triangulated** if $\mathcal{D}$ is closed under isomorphisms and translation, and for any distinguished triangle $X \to Y \to Z \to X[1]$ with $X, Y \in \operatorname{ob}(\mathcal{D})$, we also have $Z \in \operatorname{ob}(\mathcal{D})$.

In the following lemmas/propositions, $\mathcal{A}$ is an abelian category with enough injectives and $\mathcal{I}$ is the subcategory of injectives.

Lemma 5.28. *Let $X \in \operatorname{ob}(K^+(\mathcal{A})), I \in \operatorname{ob}(K^+(\mathcal{I}))$. If X is acyclic, then any morphism $X \xrightarrow{f} I$ is nullhomotopic.*

Proof. Since X, I are both bounded below, we may apply some number of translations to assume that $X^p = I^p = 0$ for $p < 0$. We need morphisms $h^p : X^p \to I^{p-1}$ for $p > 0$ such that

$$f^p = h^{p+1} d_X^p + d_I^{p-1} h^p \tag{5.1}$$

We construct the morphisms h^p inductively, starting with h^1. Since X is acyclic, d_X^0 is a monomorphism, and since I^0 is injective, there exists a morphism $h^1 : X^1 \to I^0$ such that $f^0 = h^1 d_X^0$.

$$\begin{array}{ccccc} 0 & \longrightarrow & X^0 & \xrightarrow{d_X^0} & X^1 \\ & & {\scriptstyle f^0}\downarrow & \swarrow {\scriptstyle h^1} & \\ & & I^0 & & \end{array}$$

Sinced $d_I^{-1} = 0$, equation 5.1 is true for $p = 0$. Now assume we have $h^1, h^2, \dots, h^n$ such that equation 5.1 holds for $p = 1, \dots, n-1$. We construct h^{n+1} as follows. Let

$$\phi := f^n - d_I^{n-1} h^n : X^n \to I^n$$

Then

$$\begin{aligned} \phi d_X^{n-1} &= f^n d_X^{n-1} - d_I^{n-1} h^n d_X^{n-1} \\ &= d_I^{n-1} f^{n-1} - d_I^{n-1} h^n d_X^{n-1} \\ &= d_I^{n-1} \left(f^{n-1} - h^n d_X^{n-1} \right) \\ &= d_I^{n-1} \left(d_I^{n-2} h^{n-1} \right) \\ &= 0 \end{aligned}$$

Hence ϕ factors through $\operatorname{coker} d_X^{n-1}$. Since X is acyclic, $\operatorname{coker} d_X^{n-1} = \operatorname{coim} d_X^n = \operatorname{coker}(\ker d_X^n \to X^n)$. Also, d_X^n induces a monomorphism $\operatorname{coim} d_X^n \to X^{n+1}$.

$$\begin{array}{ccccc} X^n & \longrightarrow & \operatorname{coim} d_X^n & \hookrightarrow & X^{n+1} \\ & {\scriptstyle \phi}\searrow & \downarrow {\scriptstyle \bar{\phi}} & & \\ & & I^n & & \end{array}$$

Since I^n is injective, there exists $h^{n+1} : X^{n+1} \to I^n$ such that $\phi = h^{n+1} d_X^n$.

$$\begin{array}{ccccc} 0 & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} \\ & & {\scriptstyle \phi}\downarrow & \swarrow {\scriptstyle h^{n+1}} & \\ & & I^n & & \end{array}$$

So we get $h^{n+1} d_X^n = \phi = f^n - d_I^{n-1} h^n$ which rearranges to get equation 5.1 with $p = n$. $\square$

Proposition 5.29. *Let $\mathcal{A}$ be an abelian category and $X \in \mathrm{ob}(K(^+(\mathcal{A})), I \in \mathrm{ob}(K^+(\mathcal{I}))$. For any quasi-isomorphism $s : I \to X$, there exists a morphism $t : X \to I$ such that $ts = \mathrm{Id}_X$ in $K^+(\mathcal{A})$.*

Proof. Let $s : I \to X$ be a quasi-isomorphism with I, X as above. Extend s to a distinguished triangle involving the cone of s.

$$I \xrightarrow{s} X \xrightarrow{i} C_s \xrightarrow{p} I[1]$$

Since s is a quasi-isomorphism, C_s is acyclic. Then by the previous lemma applied to p, p is nullhomotopic, so there is a nullhomotopy giving us maps

$$h^n : C_s^n = I^{n+1} \oplus X^n \to I[1]^{n-1} = I^n$$

which we can write as a matrix

$$h^n = \begin{pmatrix} k^{n+1} & t^n \end{pmatrix}$$

for some morphisms $k^{n+1} : I^{n+1} \to I^n$ and $t^n : X^n \to I^n$. The fact that this is a nullhomotopy means that

$$p^n = d_{I[1]}^{n-1} h^n + h^{n+1} d_{C_s}^n$$

so

$$\begin{aligned} \begin{pmatrix} \mathrm{Id}_{I^{n+1}} & 0 \end{pmatrix} &= \begin{pmatrix} -d_I^n \end{pmatrix} \begin{pmatrix} k^{n+1} & t^n \end{pmatrix} + \begin{pmatrix} k^{n+2} & t^{n+1} \end{pmatrix} \begin{pmatrix} -d_I^{n+1} & 0 \\ s^{n+1} & d_X^n \end{pmatrix} \\ &= \begin{pmatrix} -d_I^n k^{n+1} & -d_I^n t^n \end{pmatrix} + \begin{pmatrix} -k^{n+2} d_I^{n+1} + t^{n+1} s^{n+1} & t^{n+1} d_X^n \end{pmatrix} \end{aligned}$$

hence

$$\mathrm{Id}_I^{n+1} = -d_I^n k^{n+1} - k^{n+2} d_I^{n+1} + t^{n+1} s^{n+1}$$

and

$$d_I^n t^n = t^{n+1} d_X^n$$

The second equation says that t is a morphism of complexes $X^n \to I^n$, and the first equation says that $t^{n+1}s^{n+1}$ is homotopic to $\mathrm{Id}_{I^{n+1}}$, which is what we needed to prove. □

Proposition 5.30. *Let $I, J \in \mathrm{ob}(K^+(\mathcal{I}))$. Then any quasi-isomorphism $s : I \to J$ is an isomorphism in $K^+(\mathcal{I})$.*

Proof. By the previous proposition, there exists a morphism $t : I \to I$ such that $ts = \mathrm{Id}_I$ in $K^+(\mathcal{I})$. So taking homology

$$H^p(t) H^p(s) = \mathrm{Id}_{H^p(I)}$$

Since s is a quasi-isomorphism, $H^p(s)$ is an isomorphism, so the previous equation implies that $H^p(t)$ is also an isomorphism. Thus t is a quasi-isomorphism. Then using the previous proposition again, there exists $u : I \to J$ such that $ut = \mathrm{Id}_J$ in $K^+(\mathcal{I})$, hence $u = uts = s$ implies s is an isomorphism in $K^+(\mathcal{I})$. □

Theorem 5.31. *Let $\mathcal{A}$ be an abelian category with enough injectives. The functor*

$$K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

[18] *is an equivalence of categories.*

Proof. Theorem 5.25 says that this functor is essentially surjective, so we just need to show that it is fully faithful.

[18] This is the inclusion $K^+(\mathcal{I}) \to K^+(\mathcal{A})$ followed by $Q : K^+(\mathcal{A}) \to D^+(\mathcal{A})$.

Let S^+ be the class of quasi-isomorphisms in $K^+(\mathcal{A})$ and $S^+_{\mathcal{I}} = S^+ \cap \mathrm{Mor}(K^+(\mathcal{I}))$ be the class of quasi-isomorphisms in $K^+(\mathcal{I})$. By the previous proposition, everything in $S^+_{\mathcal{I}}$ is an isomorphism in $K^+(\mathcal{I})$. So $S^+_{\mathcal{I}}$ is a localizing class compatible with triangulation, and

$$K^+(\mathcal{I})[(S^+_{\mathcal{I}})^{-1}] = K^+(\mathcal{I})$$

since we're only localizing by isomorphisms. Also, if $I \xrightarrow{s} X$ is in $K^+(\mathcal{A})$, there exists $X \xrightarrow{t} I$ such that $ts = \mathrm{Id}_I$ where t is a morphism in $K^+(\mathcal{A})$. This implies $ts \in S^+$. Hence lemma 5.18 applies and shows that

$$K^+(\mathcal{I}) = K^+(\mathcal{I})[(S^+_{\mathcal{I}})^{-1}] \to K^+(\mathcal{A})[(S^+)^{-1}] = D^+(\mathcal{A})$$

is fully faithful. □

Proposition 5.32. *Let $I \in \mathrm{ob}(K^+(\mathcal{I}))$ and $X \in \mathrm{ob}(K(\mathcal{A}))$. The localization functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ gives a bijection*

$$\kappa : \mathrm{Hom}_{K(\mathcal{A})}(X, I) \to \mathrm{Hom}_{D(\mathcal{A})}(X, I)$$

Proof. The map κ takes a morphism in the homotopy category to the "same" morphism in $D(\mathcal{A})$. To be precise, κ takes a morphism $X \xrightarrow{f} I$ to the equivalence class of the left roof

$$\begin{array}{ccc} & X & \\ {\scriptstyle \mathrm{Id}_X}\swarrow & & \searrow{\scriptstyle f} \\ X & & I \end{array}$$

First we prove that κ is injective. Since κ is a morphism of abelian groups, it suffices to show that κ has trivial kernel. Suppose $X \xrightarrow{f} I$ such that $\kappa(f) = 0$. Then there exists a quasi-isomorphism $I \xrightarrow{s} Y$ such that $sf = 0$. We may assume $Y \in \mathrm{ob}(K^+(\mathcal{A}))$. From previous results, there exists $Y \xrightarrow{t} I$ such that $ts = \mathrm{Id}_I$ in $K^+(\mathcal{A})$. Then $tsf = f = 0$, so κ has trivial kernel, and is injective.

Now we prove κ is surjective. Let $X \xrightarrow{\phi} I$ be a morphism in $D(\mathcal{A})$. Represent ϕ as a right roof.

$$\begin{array}{ccc} & Y & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle s}\,\sim \\ X & & I \end{array}$$

with s a quasi-isomorphism. Then there exists $Y \xrightarrow{t} I$ such that $ts = \mathrm{Id}_I$ and the argument constructing t showed that t is a quasi-isomorphism. Hence ϕ is represented by the equivalent roof

$$\begin{array}{ccc} & I & \\ {\scriptstyle tf}\nearrow & & \nwarrow{\scriptstyle ts=\mathrm{Id}}\,\sim \\ X & & I \end{array}$$

hence $\phi = \kappa(tf)$. □

5.3 Extensions and Ext groups

In this section we discuss a view of Ext groups from the perspective of the derived category. As in the previous section, $\mathcal{A}$ is an abelian category with enough injectives, and $\mathcal{I}$ is the full additive subcategory of injective objects.

Definition 5.33. Let $\mathcal{A}$ be as above, with objects X, Y. We give the classical definition of $\text{Ext}^n_{\mathcal{A}}(X, Y)$. Choose an injective resolution of Y.

$$0 \to Y \to I^0 \to I^1 \to \cdots$$

and form the complex

$$0 \to \text{Hom}_{\mathcal{A}}(X, I^0) \to \text{Hom}_{\mathcal{A}}(X, I^1) \to \cdots$$

We define $\text{Ext}^n_{\mathcal{A}}(X, Y)$ to be the nth cohomology group of this latter complex.

Remark 5.34. In the classical approach, one has to then show that the choice of injective resolution for Y in the previous construction does not change the homology of the second complex, but we will omit this discussion and focus on giving an equivalent description in terms of the derived category.

Let X, Y be as above, with the chosen injective resolution of Y. Set $X[0]$ to be the complex with X in degree zero, and analogously $Y[0]$ is the complex with Y in degree zero.

$$\begin{aligned} X[0] = &\qquad \cdots \to 0 \to X \to 0 \to \cdots \\ Y[0] = &\qquad \cdots \to 0 \to Y \to 0 \to \cdots \end{aligned}$$

Let I be the following complex in $K^+(\mathcal{I})$.

$$I = \qquad \cdots \to 0 \to I^0 \to I^1 \to \cdots$$

The map $Y \to I^0$ gives a quasi-isomorphism of complexes $Y[0] \to I$.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

which is then an isomorphism in $D^+(\mathcal{A})$. Hence

$$\text{Hom}_{D(\mathcal{A})}(X[0], Y[n]) \cong \text{Hom}_{D(\mathcal{A})}(X[0], I[n])$$

where $Y[n]$ is the complex $Y[0]$ shifted n times, so it has Y in degree $-n$ and zeros elsewhere. Then from proposition 5.32 we have

$$\text{Hom}_{D(\mathcal{A})}(X[0], I[n]) \cong \text{Hom}_{K(\mathcal{A})}(X[0], I[n])$$

Theorem 5.35. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any $X, Y \in \text{ob}(\mathcal{A})$,*

$$\text{Ext}^n_{\mathcal{A}}(X, Y) = \text{Hom}_{D(\mathcal{A})}(X[0], Y[n])$$

Proof. $\mathrm{Hom}_{K(\mathcal{A})}(X[0], I[n])$ consists of homotopy class of morphisms of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1} & \longrightarrow & \cdots \end{array}$$

Note that a morphism $X \xrightarrow{f} I^n$ in $\mathcal{A}$ gives rise to such a morphism of complexes as above if and only if the composition $X \xrightarrow{f^0} I^n \xrightarrow{d_I^n} I^{n+1}$ is zero. So

$$\mathrm{Hom}_{C(\mathcal{A})}(X[0], I[n]) = \ker\Big(\mathrm{Hom}_{\mathcal{A}}(X, I^n) \to \mathrm{Hom}_{\mathcal{A}}(X, I^{n+1})\Big)$$

the map on the right being given by composition with d_I^n. Also, $X[0] \xrightarrow{f} I[n]$ is nullhomotopic if and only if there exists a morphism $X \xrightarrow{h} I^{n-1}$ in $\mathcal{A}$ such that $f^0 = d_I^{n-1}h$, which is to say, if and only if the composition $X \xrightarrow{h} I^{n-1} \xrightarrow{d_I^{n-1}} I^n$ is f^0. Thus the subgroup of $\mathrm{Hom}_{C(\mathcal{A})}(X[0], I[n])$ of nullhomotopic moprphisms is the image of

$$\mathrm{Hom}_{\mathcal{A}}(X, I^{n-1}) \to \mathrm{Hom}_{\mathcal{A}}(X, I^n) \qquad h \mapsto d_I^{n-1}h$$

Thus $\mathrm{Hom}_{K(\mathcal{A})}(X[0], I[n])$ is precisely this quotient, which is coincides with our earlier definition of $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$.

$$\mathrm{Hom}_{K(\mathcal{A})}(X[0], I[n]) = \frac{\ker\Big(\mathrm{Hom}_{\mathcal{A}}(X, I^n) \to \mathrm{Hom}_{\mathcal{A}}(X, I^{n+1})\Big)}{\mathrm{im}\Big(\mathrm{Hom}_{\mathcal{A}}(X, I^{n-1}) \to \mathrm{Hom}_{\mathcal{A}}(X, I^n)\Big)}$$

□

Remark 5.36. The same argument as above (or just applying translation) gives a more general statement:

$$\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{D(\mathcal{A})}(X[m], Y[n+m])$$

for any $m \in \mathbb{Z}$.

Definition 5.37. Let $A, B \in \mathrm{ob}(\mathcal{A})$. Define an abelian group $\mathrm{Ext}_{\mathcal{A}}(A, B)$ whose elements are equivalence classes of extensions

$$0 \to B \to E \to A \to 0$$

where two extensions are equivalent if there is an isomorphism of short exact sequences as below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Addition in $\mathrm{Ext}_{\mathcal{A}}(A, B)$ is by Baer sum, which is essentially taking the pullback of E, E', but we omit the details of this at the moment.

Definition 5.38. An extension $0 \to B \to E \to A \to 0$ can be viewed as a quasi-isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

To be clear, we're putting the A and E in degree zero, and B in degree -1. We denote $A[0]$ as before the complex with A in degree zero, $B[1]$ as the complex with just B in degree -1, and let $E(A, B)$ be the complex

$$0 \to B \to E \to 0$$

with degrees as designated above. Let $E(A, B) \xrightarrow{s} A[0]$ be the quasi-isomorphism depicted above, and let $E(A, B) \xrightarrow{f} B[1]$ be the identity on the B component and zero elsewhere. Then we can take the equivalence class of the left roof

$$\begin{array}{ccc} & E(A,B) & \\ {\scriptstyle s}\swarrow\sim & & \searrow{\scriptstyle f} \\ A[0] & & B[1] \end{array}$$

and obtain a morphism $Q(f)Q(s)^{-1} \in \mathrm{Hom}_{D(\mathcal{A})}(A[0], B[1]) = \mathrm{Ext}^1_{\mathcal{A}}(A, B)$. In order to have this give rise to a map $\mathrm{Ext}_{\mathcal{A}}(A, B) \to \mathrm{Ext}^1_{\mathcal{A}}(A, B)$, we need to show that equivalent extensions give rise to the same morphism in the derived category. If we have an equivalence of extensions given by an isomorphism $\epsilon : E \to E'$, then it gives rise to an equivalence of roofs

$$\begin{array}{ccccc} & & E(A,B) & & \\ & {\scriptstyle \mathrm{Id}}\swarrow & & \searrow{\scriptstyle (\mathrm{Id}_B,\epsilon)} & \\ & E(A,B) & & E'(A,B) & \\ {\scriptstyle s}\swarrow & & {\scriptstyle s'}\ \times\ {\scriptstyle f} & & \searrow{\scriptstyle f'} \\ A[0] & & & & B[1] \end{array}$$

So this does give rise to a well defined map

$$\Phi : \mathrm{Ext}_{\mathcal{A}}(A, B) \to \mathrm{Ext}^1_{\mathcal{A}}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A[0], B[1])$$

Theorem 5.39. *The map Φ defined above is an isomorphism of abelian groups.*

Proof. We omit everything except a sketch of surjectivity, since that's the most interesting part. Given $\phi \in \mathrm{Hom}_{D(\mathcal{A})}(A[0], B[1])$, represent it by a left roof

$$\begin{array}{ccc} & L & \\ {\scriptstyle s}\swarrow\sim & & \searrow{\scriptstyle f} \\ A[0] & & B[1] \end{array}$$

We can then replace L by its truncation $\tau_{\leq 0}(L)$ and assume $L^p = 0$ for $p > 0$, since L is quasi-isomorphic to A. Also, $H^0(L) = A$, and $H^i(L) = 0$ for $i \neq 0$. Then we have a diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & L^{-2} & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^{-1}} & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

with the top row exact. Let $E(\phi) = \operatorname{coker}(L^{-1} \to L^0 \oplus B) = \operatorname{coker}(d_L^{-1}, f^{-1})$. You can check that this gives an extension $0 \to B \to E(\phi) \to A \to 0$ which satisfies $\Phi(E(\phi)) = \phi$. □

5.4 Derived functors

In this section we discuss the conditions needed to have a functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories induce a functor on the derived categories $D^*(\mathcal{A}) \to D^*(\mathcal{B})$. First, we start out only needing the assumption that $\mathcal{A}, \mathcal{B}$ are additive, and see that F always induces, rather naively, a functor on the homotopy categories.

Definition 5.40. Let $\mathcal{A}, \mathcal{B}$ be additive categories, and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. Applying F term-by-term gives an additive functor $C(F) : C(\mathcal{A}) \to C(\mathcal{B})$ on the chain complex categories. This functor $C(F)$ takes homotopic chain maps in $C(\mathcal{A})$ to homotopic chain maps in $C(\mathcal{B})$, so it induces $K(F) : K(\mathcal{A}) \to K(\mathcal{B})$.

On objects, $K(F)$ just applies F term-by-term, and on morphisms, $K(F)$ applies F term-by-term to a morphism $f^n : X^n \to Y^n$, and this respects homotopy classes by the previous remark. Similarly, if we take a bounded version of the homotopy categories we get $K(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$, where $*$ can be $+, -, b$.

Remark 5.41. The functor $K(F)$ above literally commutes with the translation functors on $K(\mathcal{A}), K(\mathcal{B})$, not just commute up to natural isomorphism.

$$K(F) \circ T_{K(\mathcal{A})} = T_{K(\mathcal{B})} \circ K(F)$$

Lemma 5.42. *The functor $C(F)$ preserves cones of morphisms. That is, if $X \xrightarrow{f} Y$ is a morphism of complexes in $C^*(\mathcal{A})$ with cone C_f, then $C(F)$ applied to the cone C_f is the cone of $C(F)(f)$.*

$$C(F)(C_f) = C_{C(F)(f)}$$

Proof. Since F is additive, it commutes with finite biproducts. So for $p \in \mathbb{Z}$, the pth term of the cone $C_{C(F)(f)}$ is

$$C^p_{C(F)(f)} = F(X)^{p+1} \oplus F(Y)^p \cong F(X^{p+1} \oplus Y^p) = C(F)(C^p_f)$$

Thus the pth object term of the cone $C_{C(F)(f)}$ is isomorphic to the pth term of the image of the cone C_f. Similarly, the pth boundary morphisms agree by the following calculation.

$$d^p_{C(F)(f)} = \begin{pmatrix} -d^{p+1}_{C(F)(X)} & 0 \\ C(F)(f)^{p+1} & d^p_{C(F)(Y)} \end{pmatrix} = \begin{pmatrix} -F(d^{p+1}_X) & 0 \\ F(f^{p+1}) & F(d^p_Y) \end{pmatrix} = F(d^p_{C_f})$$

□

Corollary 5.43. *$K(F)$ takes standard distinguished triangles in $K^*(\mathcal{A})$ (those given by cones) to distinguished triangles in $K^*(\mathcal{B})$. Thus $K(F)$ takes all distinguished triangles in $K^*(\mathcal{A})$ to distinguished triangles in $K^*(\mathcal{B})$.*

Corollary 5.44. *$K(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$ is exact.*

Proof. We already noted it commutes with translation, and the previous corollary says that it preserves distinguished triangles. □

To summarize the previous discussion:

Proposition 5.45. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Applying F term-by-term gives an exact functor of triangulated categories $K(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$.*

Now we want to see if we can pass to the derived category. Assume $\mathcal{A}, \mathcal{B}$ are abelian. Let $Q_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A})$ and $Q_{\mathcal{B}} : K^*(\mathcal{B}) \to D^*(\mathcal{B})$ be the respective localization functors. We would like to have a functor $D^*(\mathcal{A}) \to D^*(\mathcal{B})$ making the following diagram commute.

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{K(F)} & K^*(\mathcal{B}) \\ \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ D^*(\mathcal{A}) & \dashrightarrow & D^*(\mathcal{B}) \end{array}$$

A natural strategy would be to use the universal property given by (5) in proposition 4.50 to obtain this functor. However, to apply this universal property, we would need to know that $K(F)$ takes quasi-isomorphisms in $K^*(\mathcal{A})$ to quasi-isomorphisms in $K^*(\mathcal{B})$, which is not true, in general.[19]

As a first illustration of how this might happen, we can say that at least in a somewhat trivial case, this happens, which is when F is exact. Of course, if F is exact, then there isn't really much homology going to begin with, so this is not the most interesting case. Despite this, we record the following fact.

Proposition 5.46. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then for any quasi-isomorphism $X \xrightarrow{s} Y$ in $K^*(\mathcal{A})$, $K(F)(s)$ is a quasi-isomorphism in $K^*(\mathcal{B})$.*

Proof. Recall that s is a quasi-isomorphism if and only if the cone C_s is acyclic. By lemma 5.42, $C_{C(F)(s)} = C(F)(C_s)$. Since s is a quasi-isomorphism, C_s is acyclic. Since F is exact, it takes the acyclic complex C_s to an acyclic complex $C(F)(C_s)$. Then $C_{C(F)(s)}$ is also acyclic, so $C(F)(s)$ is a quasi-isomorphism. □

Corollary 5.47. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then there exists a unique exact functor $D(F) : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ such that the following diagram commutes.*

[19] I don't know a specific counterexample, unfortunately. Probably the place to look for a counterexample is just to take $F : \mathcal{A} \to \mathcal{B}$ to be some functor which is neither left nor right exact.

$$
\begin{array}{ccc}
K^*(\mathcal{A}) & \xrightarrow{K(F)} & K^*(\mathcal{B}) \\
\downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\
D^*(\mathcal{A}) & \dashrightarrow & D^*(\mathcal{B})
\end{array}
$$

Proof. By the previous proposition, $K(F)$ takes quasi-isomorphisms in $K^*(\mathcal{A})$ to quasi-isomorphisms in $K^*(\mathcal{B})$. Then we can apply the universal property 4.50 to the functor $Q_{\mathcal{B}} \circ K(F) : K^{(}\mathcal{A}) \to D^{(}\mathcal{B})$ to obtain a unique additive functor $D(F) : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ making the diagram commute. □

Next our goal is to obtain a functor on the derived categories under a weaker assumption than exactness of F, since this is a very strict requirement. For the moment, we conduct the discussion in a slightly more general setting of arbitrary triangulated categories rather than the homotopy category of an abelian category.

Remark 5.48. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories, $F : \mathcal{C} \to \mathcal{D}$ be an exact functor compatible with triangulation, and S a localizing class in $\mathcal{C}$ compatible with triangulation. If $F(s)$ is an isomorphism for every $s \in S$, then the universal property just applied above gives a unique exact functor $\overline{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = \overline{F}Q$, where $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is the localization functor.

Definition 5.49. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories, $F : \mathcal{C} \to \mathcal{D}$ be an exact functor, and S a localizing class in $\mathcal{C}$ compatible with triangulation. A **right derived functor** of F is a pair (RF, ϵ_F) where $RF : \mathcal{C}[S^{-1}] \to \mathcal{D}$ is an exact functor, and $\epsilon_F : F \to RF \circ Q$ is a graded natural transformation, and (RF, ϵ_F) satisfy the following universal property:

Given an exact functor $G : \mathcal{C}[S^{-1}] \to \mathcal{D}$ and a graded natural transformation $\epsilon_G : F \to G \circ Q$, there exists a unique graded natural transformation $\eta : RF \to G$ such that the following diagram commutes.

$$
\begin{array}{ccc}
 & F & \\
{\scriptstyle \epsilon_F}\swarrow & & \searrow{\scriptstyle \epsilon_G} \\
RF \circ Q & \underset{\eta'}{\dashrightarrow} & G \circ Q
\end{array}
$$

where η' is the natural transformation given by

$$\eta'_X = \eta_{QX} : RF \circ Q(X) \to G \circ Q(X)$$

for an object $X \in \mathrm{ob}(\mathcal{C})$.

Remark 5.50. To keep everything in the previous definition straight, we include the following table.

Variable	Type of object
$\mathcal{C}, \mathcal{D}$	Category
$F, RF \circ Q, G \circ Q$	Functor $\mathcal{C} \to \mathcal{D}$
Q	Functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$
RF, G	Functor $\mathcal{C}[S^{-1}] \to \mathcal{D}$
ϵ_F	Natural transformation $F \to RF \circ Q$
ϵ_G	Natural transformation $F \to G \circ Q$
η	Natural transformation $RF \to G$
η'	Natural transformation $RF \circ Q \to G \circ Q$

In particular, the distinction between η and η' is somewhat subtle. The fact that η is a natural transformation $RF \to G$ means that for every object $Y \in \mathrm{ob}(\mathcal{C}[S^{-1}])$, there is a morphism $\eta_Y \in \mathrm{Hom}_{\mathcal{D}}(RFY, GY)$, subject to a commutativity condition in $\mathcal{D}$.

Then, η' being a natural transformation $RF \circ Q \to G \circ Q$ means that for every object $X \in \mathrm{ob}(\mathcal{C})$, there is a morphism $\eta'_Y \in \mathrm{Hom}_{\mathcal{D}}(RF \circ Q(X), G \circ Q(X))$. By definition of η', for $Y = Q(X)$, these are the same morphism. The fact that η_Y is defined for objects Y in $\mathcal{C}[S^{-1}]$ which are not of the form $Q(X)$ for an object X of $\mathcal{C}$ doesn't matter for what's going on in this definition.

Remark 5.51. As usual in category theory, one can reverse arrows and obtain another sensible definition. In this case, one defines left derived functors (LF, ϵ_F) in this way. Alternately, one can just talk about right derived functors in the opposite category.

Remark 5.52. Because of the universal property, a right or left derived functor is unique up to isomorphism, if it exists. In this case, "unique up to isomorphism" means that the functor RF is unique up to natural isomorphism.

Remark 5.53. If $F(s)$ is an isomorphism for all $s \in S$, the both the right derived functor RF and left derived functor LF coincide with the functor $\overline{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ of remark 5.48.

As we will see, it turns out that we can construct RF assuming something weaker than that $F(s)$ is an isomorphism for every $s \in S$. Instead, we'll only have to assume that $F(s)$ is an isomorphism for every s in some subclass S'.

Definition 5.54. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories, and let S be a localizing class in $\mathcal{C}$ compatible with triangulation. A full triangulated subcategory $\mathcal{E} \subset \mathcal{C}$ is **right adapted to** F if

(RA1) $S_{\mathcal{E}} = S \cap \mathrm{Mor}(\mathcal{E})$ is a localizing class in $\mathcal{E}$.

(RA2) For any $X \in \mathrm{ob}(\mathcal{C})$, there exists $M \in \mathrm{ob}(\mathcal{E})$ and a morphism $X \xrightarrow{s} M$ with $s \in S$.

(RA3) For any $s \in S_{\mathcal{E}}$, $F(s)$ is an isomorphism in $\mathcal{D}$.

Theorem 5.55. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories and S a localizing class in $\mathcal{C}$ compatible with triangulation. If there exists a full subcategory $\mathcal{E}$ of $\mathcal{C}$ right adapted to F, then the right derived functor (RF, ϵ_F) exists.*

Proof. We sketch the construction, and leave out the long tedious verifications. By (RA1), $S_{\mathcal{E}}$ is a localizing class compatible with triangulation so we can consider the localization $\mathcal{E}[S_{\mathcal{E}}^{-1}]$. Then using (RA2) and considerations similar to lemma 5.18, the natural functor

$$\Psi : \mathcal{E}[S_{\mathcal{E}}^{-1}] \to \mathcal{C}[S^{-1}]$$

is fully faithful. Since this functor is the identity on objects, it allows us to think of $\mathcal{E}[S_{\mathcal{C}}^{-1}]$ as a fully subcategory of $\mathcal{C}[S^{-1}]$. (RA2) also implies that Ψ is essentially surjective, so it is an equivalence of categories. Let

$$\Phi : \mathcal{C}[S^{-1}] \to \mathcal{E}[S_{\mathcal{E}}^{-1}]$$

be a pseudo-inverse for Ψ, and in particular choose Φ so that $\Phi \circ \Psi$ is the identity on $\mathrm{Id}_{\mathcal{E}[S_{\mathcal{E}}^{-1}]}$ (this is mostly a notational convenience). By (RA3), $F(s)$ is an isomorphism for $s \in S_{\mathcal{E}}$, so the universal property in proposition 4.50 applies to give a functor $\overline{F} : \mathcal{C}[S_{\mathcal{E}}^{-1}] \to \mathcal{D}$ such that $\overline{F} \circ Q_{\mathcal{E}} = F|_{\mathcal{E}}$.

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{F} & \mathcal{D} \\
\uparrow & & \uparrow \overline{F} \\
\mathcal{E} & \xrightarrow{Q_{\mathcal{E}}} & \mathcal{E}[S_{\mathcal{E}}^{-1}]
\end{array}$$

Set $RF = \overline{F} \circ \Phi$, which is a functor $\mathcal{C}[S^{-1}] \to \mathcal{D}$. Next we sketch the construction of the graded natural transformation $\epsilon_F : F \to RF \circ Q_{\mathcal{C}}$. Because Ψ, Φ are pseudo-inverse equivalences, there is a natural isomorphism

$$\beta : \mathrm{Id}_{\mathcal{C}[S^{-1}]} \to \Psi \circ \Phi$$

which means that for each $X \in \mathrm{ob}(\mathcal{C}[S^{-1}])$, there is an isomorphism $\beta_X : X \to \Psi\Phi X$ in $\mathcal{C}[S^{-1}]$. Objects of $\mathcal{C}[S^{-1}]$ are just objects of $\mathcal{C}$, and Ψ is the identity on objects, so we can think of this as an isomorphism (still in $\mathcal{C}[S^{-1}]$) $\beta : X : X \to \Phi X$. Represent β_X by a right roof.

$$\begin{array}{ccccc}
 & & K & & \\
 & \overset{f}{\nearrow} & & \overset{s}{\underset{\sim}{\nwarrow}} & \\
X & & & & \Phi X
\end{array}$$

with $K \in \mathrm{ob}(\mathcal{C})$ and $s \in S$. By (RA2), there exists a moprhism $K \xrightarrow{u} M$ with $u \in S$, $M \in \mathrm{ob}(\mathcal{E})$, so the roof above is equivalent to

$$\begin{array}{ccccc}
 & & M & & \\
 & \overset{uf}{\nearrow} & & \overset{us}{\underset{\sim}{\nwarrow}} & \\
X & & & & \Phi X
\end{array}$$

So we may as well just assume that $K \in \mathrm{ob}(\mathcal{E})$. Then since ΦX and M are in $\mathrm{ob}(\mathcal{E})$, $s \in S_{\mathcal{E}}$. So $F(s)$ is an isomorphism in $\mathcal{D}$, and we can consider $F(s)^{-1}$. Define

$$\rho_X := F(s)^{-1} \circ F(f) : FX \to F\Phi X = (RF \circ Q)(X)$$

Then one checks the following:

1. ρ_X is independent of the choice of roof representing β_X
2. The collection ρ_X defines a natural transformation $\rho = \epsilon_F : F \to RF \circ Q$
3. ϵ_F is a graded natural transformation
4. ϵ_F satisfies the universal property of the theorem

We omit these details, since they are somewhat lengthy and involved. □

Now that we have a general statement in the abstract setting of triangulated categories, we specialize toward looking at right derived functors for the homotopy and derived categories of an abelian category.

Definition 5.56. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. By a **right derived functor of** F we mean a right derived functor in the previous sense of $Q_{\mathcal{B}} \circ K(F)$.

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{K(F)} & K^*(\mathcal{B}) \\ {\scriptstyle Q_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ D^*(\mathcal{A}) & & D^*(\mathcal{B}) \end{array}$$

That is, a right derived functor of F is a pair (RF, ϵ_F) where $RF : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ is a functor and $\epsilon_F : Q_{\mathcal{B}} \circ K(F) \to RF \circ Q_{\mathcal{A}}$ is a graded natural transformation satisfying the universal property: given an exact functor $G : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ and a graded natural transformation $\epsilon_G : Q_{\mathcal{B}} \circ K(F) \to G \circ Q_{\mathcal{A}}$ there exists a unique graded natural transformation $\eta : RF \to G$ such that the following diagram commutes.

$$\begin{array}{ccccc} & & Q_{\mathcal{B}} \circ K(F) & & \\ & {\scriptstyle \epsilon_F}\swarrow & & \searrow{\scriptstyle \epsilon_G} & \\ RF \circ Q_{\mathcal{A}} & & \xrightarrow{\quad \eta' \quad} & & G \circ Q_{\mathcal{A}} \end{array}$$

Remark 5.57. If $\mathcal{R} \subset K^*(\mathcal{A})$ is a full triangulated subcategory and $S^* \subset K^*(\mathcal{A})$ is the usual localizing class of quasi-isomorphisms, then $S^*_{\mathcal{R}} := S^* \cap \mathrm{Mor}(\mathcal{R})$ is a localizaing class compatible with triangulation. In other words, when speaking of subcategories of $K^*(\mathcal{A})$ adapted to functors, we can omit (RA1).

Definition 5.58. $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. A full triangulated subcategory $\mathcal{R} \subset K^*(\mathcal{A})$ is **right adapted to** F if

(RA2) For any $X \in \mathrm{ob}(K^*(\mathcal{A}))$, there exists $M \in \mathrm{ob}(\mathcal{R})$ and a quasi-isomorphism $X \xrightarrow{s} M$.

(RA3') For any acyclic complex $M \in \mathrm{ob}(K^*(\mathcal{R}))$, the complex $K(F)(M)$ is acyclic in $K^*(\mathcal{B})$.

Using the fact that a morphism is a quasi-isomorphism if and only if the cone is acyclic, (RA3') implies (RA3), though note that the converse may not be true, so (RA3) and (RA3') are not necessarily equivalent.

This gives us a version of theorem 5.55 focused more on derived categories.

Theorem 5.59. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume there exists a full triangulated subcategory $\mathcal{R} \subset K^*(\mathcal{A})$ right adapted to F. Then there exists a right derived functor (RF, ϵ_F).*

Proof. Immediate consequence of theorem 5.55 and definition 5.58. □

We'd like to rephrase this slightly so that the conditions have less focus on the homotopy category and more on the original category $\mathcal{A}$, since such a condition will be easier to verify in practice. So we once again retool our definition of right adapted. We can't entirely get away from the homotopy category, unfortunately.

Definition 5.60. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. A fully subcategory $\mathcal{R} \subset \mathcal{A}$ is **right adapted to** F if

(A1) $\mathcal{R}$ is an additive subcategory of $\mathcal{A}$, meaning $0 \in \mathcal{R}$ and $\mathcal{R}$ is closed under finite products/coproducts.

(A2) Every object in $\mathcal{A}$ is a subobject of an object of $\mathcal{R}$. That is, for every $M \in \text{ob}(\mathcal{A})$, there exists a monomorphism $M \to R$ with $R \in \text{ob}(\mathcal{R})$.

(A3) If $R \in \text{ob}(K^+(\mathcal{R}))$ is acyclic, then $K(F)(R)$ is acyclic in $K^+(\mathcal{B})$.

Remark 5.61. In the above definition, if $R \xrightarrow{f} T$ is a morphism in $K^+(\mathcal{R})$, since $\mathcal{R}$ is closed under coproducts the cone C_f also lies in $K^+(\mathcal{R})$. Thus $K^+(\mathcal{R})$ is a full triangulated subcategory of $K^+(\mathcal{A})$.

Remark 5.62. In the above definition, $(A2)$ implies that for any complex $X \in \text{ob}(K^+(\mathcal{A}))$, there exists a quasi-isomorphism $X \xrightarrow{s} R$ for some $R \in \text{ob}(K^+(\mathcal{R}))$.

Now we can rephrase theorem 5.59 as

Theorem 5.63. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume there is a subcategory $\mathcal{R} \subset \mathcal{A}$ which is right adapted to F. Then there exists a right derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*

Now we specialize even further, to the most common adapted subcategory, that of injective objects.

Lemma 5.64. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{I}$ be the full subcategory of injective objects.*

1. *$\mathcal{I}$ satisfies (A3).*
2. *If $\mathcal{A}$ has enough injectives, then for any additive functor $F : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ abelian, $\mathcal{I}$ is right adapted to $\mathcal{I}$.*

Proof. (1) Let $I \in \text{ob}(K^+(\mathcal{I}))$ be acyclic. By lemma 5.28, the identity morphism $I \to I$ is nullhomotopic. So $I \cong 0$ in $K^+(\mathcal{A})$. So $K(F)(I) \cong 0$ in $K^+(\mathcal{B})$, so $K(F)(I)$ is acyclic.

(2) $\mathcal{I}$ satisfies (A1), and (A3) by part (1). Condition (A2) is equivalent to the fact that $\mathcal{A}$ has enough injectives. □

Theorem 5.65. *Let $\mathcal{A}$ be an abelian category with enough injectives, and $F : \mathcal{A} \to \mathcal{B}$ be an addtive functor to an abelian category $\mathcal{B}$. Then F has a right derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*

Proof. Immediate from theorem 5.59 and the previous lemma. □

Remark 5.66. We summarize and describe the previous theorem more concretely. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $\mathcal{I}$ be the full subcategory of injective objects. Recall that we have an equivalence of categories

$$K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

which is the obvious functor, just the inclusion into $K^+(\mathcal{A})$ followed by the localization functor. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Tracing back to the construction of the right derived functor in theorem 5.55, we can describe the constructed functor in theorem 5.65 as follows: it is the composition

$$D^+(\mathcal{A}) \to K^+(\mathcal{I}) \xrightarrow{K(F)} K^+(\mathcal{B}) \xrightarrow{Q_\mathcal{B}} D^+(\mathcal{B})$$

where the first arrow is a pseudo-inverse for the equivalence of categories mentioned above. Next, we want to relate our description of right derived functors to the classical construction of right derived functors using injective resolutions.

Definition 5.67. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, where $\mathcal{A}$ has enough injectives. Let $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be the right derived functor of theorem 5.65. Recall the functor

$$D_\mathcal{A} : \mathcal{A} \to D^+(\mathcal{A}) \qquad X \mapsto \cdots \to 0 \to X \to 0 \to \cdots$$

Let $T_{D^+(\mathcal{B})}$ be the translation functor on $D^+(\mathcal{B})$, and $H^n : D^+(\mathcal{A}) \to \mathcal{A}$ be the cohomology functor. For $n \in \mathbb{Z}$, define

$$R^nF : \mathcal{A} \to \mathcal{B} \qquad R^nF = H^n \circ RF \circ D_\mathcal{A} = H^0 \circ T^n_{D^+(\mathcal{B})} \circ RF \circ D_\mathcal{A}$$

Remark 5.68. We'll show in a minute that this functor R^nF coincides with the classical right derived functors of F in the case where F is left exact. Before we get to the proof, a few notes of setup. Recall that by construction of RF, we have a natural transformation

$$\epsilon_F : Q_\mathcal{B} \circ K(F) \to RF \circ Q_\mathcal{A}$$

The functors $Q_\mathcal{B} \circ K(F)$ and $RF \circ Q_\mathcal{A}$ are functors $K(\mathcal{A}) \to D^+(\mathcal{B})$. Let $M \in \mathrm{ob}(\mathcal{A})$. Then consider $K_\mathcal{A}M \in \mathrm{ob}(K(\mathcal{A}))$, which is just M in degree zero. The ϵ_F gives a morphism in $D^+(\mathcal{B})$.

$$\epsilon_{F,K_\mathcal{A}M} : \Big(Q_\mathcal{B} \circ K(F)\Big)(K_\mathcal{A}M) \to \Big(RF \circ Q_\mathcal{A}\Big)(K_\mathcal{A}M)$$

We can describe the source and target of this in simpler terms. Since $K(F)$ just applies F term-by-term to complexes,

$$\Big(Q_\mathcal{B} \circ K(F)\Big)(K_\mathcal{A}M) = D_\mathcal{B}(FM)$$

and

$$\Big(RF \circ Q_{\mathcal{A}}\Big)\big(K_{\mathcal{A}}M\big) = RF(D_{\mathcal{A}}M)$$

So we can think of $\epsilon_{F,K_{\mathcal{A}}M}$ as a morphism

$$\epsilon_{F,K_{\mathcal{A}}M} : D_{\mathcal{B}}(FM) \to RF(D_{\mathcal{A}}M)$$

Then if we apply the H^0 cohomology functor, we obtain

$$H^0(\epsilon_{F,K_{\mathcal{A}}M} : H^0(D_{\mathcal{B}}(FM)) \to H^0(RF(D_{\mathcal{A}}M))$$

Since $D_{\mathcal{B}}(FM)$ is concentrated in degree zero, the source here is just FM. And the target is, by definition, $R^0F(M)$. So we have

$$H^0(\epsilon_{F,K_{\mathcal{A}}M} : FM \to R^0F(M)$$

Since ϵ_F is a natural transformation, it makes an appropriate diagram commute. Applying H^0 to that diagram shows that $H^0(\epsilon_F)$ is a natural transformation

$$H^0(\epsilon_F) : F \to R^0F$$

which is given on an object $M \in \mathrm{ob}(\mathcal{A})$ by $H^0(\epsilon_{F,K_{\mathcal{A}}M}$ above.

With the preceeding remark out of the way, we can describe properties of our constructed functors R^nF, in particular that they coincide with the classical right derived functors when F is left exact.

Proposition 5.69. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, with $\mathcal{A}$ having enough injectives.*

(1) $R^nF = 0$ for $n < 0$.

(2) R^0F is left exact.

(3) The natural transformation

$$H^0(\epsilon_F) : F \to R^0F$$

is a natural isomorphism if and only if F is left exact.

(4) For any short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in $\mathcal{A}$, there is a long exact sequence

$$0 \to R^0F(L) \xrightarrow{R^0F(f)} R^0F(M) \xrightarrow{R^0F(g)} R^0F(N) \to R^1F(L) \xrightarrow{R^1F(f)} R^1F(M) \to \cdots$$

(5) Let $M \in \mathrm{ob}(\mathcal{A})$, and let $0 \to M \to I^0 \to I^1 \to \cdots$ be any injective resolution of M. Let I be the complex $\cdots \to 0 \to I^0 \to I^1 \to \cdots$. Then for all $n \geq 0$,

$$R^nF(M) \cong H^n\Big(C(F)(I)\Big)$$

Before the proof, we note that combining (2), (3), and (5) gives the assertion made previously, that R^nF as defined here coincides with the classical description provided that F is left exact.

Proof. We start with (4). Let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathcal{A}$. Applying $D_{\mathcal{A}}$, we get a distinguished triangle in $D^+(\mathcal{A})$.

$$D_{\mathcal{A}}(L) \to D_{\mathcal{A}}(M) \to D_{\mathcal{A}}(N) \to D_{\mathcal{A}}(L)[1]$$

Since RF is exact, applying it gives a distinguished triangle in $D^+(\mathcal{B})$.

$$RF \circ D_{\mathcal{A}}(L) \to RF \circ D_{\mathcal{A}}(M) \to RF \circ D_{\mathcal{A}}(N) \to RF \circ D_{\mathcal{A}}(L)[1]$$

Since H^0 is a cohomological functor, it takes distinguished triangles to long exact sequences. So we obtain a long exact sequence (in $\mathcal{B}$)

$$\cdots \to H^n \circ RF \circ D_{\mathcal{A}}(L) \to H^n \circ RF \circ D_{\mathcal{A}}(M) \to H^n \circ RF \circ D_{\mathcal{A}}(N) \to H^{n+1} \circ RF \circ D_{\mathcal{A}}(L) \to \cdots$$

which by definition of R^nF is the same as

$$\cdots \to R^nF(L) \to R^nF(M) \to R^nF(N) \to R^{n+1}F(L) \to \cdots$$

This isn't quite the long exact sequence we want, since in (4) the terms for $n < 0$ vanish, but this will be immediate once we've proved (1). So (4) is mostly done.

Next we prove (5). Let $M \in \text{ob}(\mathcal{A})$, and let $0 \to M \to I^0 \to I^1 \to \cdots$ be an injective resolution of M, which we think of as a quasi-isomorphism $D_{\mathcal{A}}M \to I$, where I is as in the statement of (5). That is, $D_{\mathcal{A}}M \cong I$ in $D^+(\mathcal{A})$. So applying RF, we get an isomorphism in $D^+(\mathcal{B})$.

$$RF \circ D_{\mathcal{A}}(M) \cong RF(I)$$

Recalling the construction of RF as the composition of various functors as in remark 5.66, we see that $RF(I) = K(F)(I)$, and $K(F)(I)$ is essentially $C(F)(I)$, just viewed in a different category. So then

$$R^nF(M) = H^n \circ RF \circ D_{\mathcal{A}}(M) \cong H^n \circ RF(I) = H^n\Big(C(F)(I)\Big)$$

This finishes the proof of (5). Then (1) is immediate from (5), since $H^n(C(F)(I)) = 0$ for $n < 0$, as the terms of I vanish for $n < 0$. As noted above, having (1) also completes the proof of (4).

Next we can easily prove (2) as an immediate consequence of (4). By (4), the short exact sequence $0 \to L \to M \to N \to 0$ gives rise to an exact sequence $0 \to R^0F(L) \to R^0F(M) \to R^0F(N)$, which is exactly the condition that R^0F is left exact.

Now we just sketch a proof of (3). One direction at least is easy. If $H^0(\epsilon_F)$ is an isomorphism, then $F \cong R^0F$ which is left exact by (2), so F is left exact.

Conversely, suppose F is left exact. We need to show that $H^0(\epsilon_F)$ is a natural isomorphism. Take an injective resolution $0 \to M \to I^0 \to I^1 \to \cdots$. Then

$$0 \to FM \to FI^0 \to FI^1$$

is exact, and tracing carefully through the definitions shows that

$$H^0(\epsilon_{F,K_{\mathcal{A}}M}) : FM \to R^0FM \cong H^0(K(F)(I)) \cong FM$$

is an isomorphism, but we omit the details. □

We quickly sketch the dual definition and results for left derived functors. Nothing here is surprising, it all simply dualizes the right derived functor statements.

Definition 5.70. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. A full subcategory $\mathcal{R} \subset \mathcal{A}$ is **left adapted to** F if

(LA1) $\mathcal{R}$ is an additve subcategory of $\mathcal{A}$, meaning $0 \in \mathcal{R}$ and $\mathcal{R}$ is closed under finite products/coproducts

(LA2) Every object in $\mathcal{A}$ is a quotient of an object of $\mathcal{R}$. That is, for every $M \in \text{ob}(\mathcal{A})$, there exists an epimorphism $R \to M$ with $R \in \text{ob}(\mathcal{R})$.

(LA3) If $R \in \text{ob}(K^-(\mathcal{R}))$ is acyclic, then $K(F)(R)$ is acyclic in $K^-(\mathcal{B})$.

Theorem 5.71. *Let $\mathcal{A}$ be an abelian category with enough projectives. Any additive functor $F : \mathcal{A} \to \mathcal{B}$ has a left derived functor $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$.*

Definition 5.72. For $n \in \mathbb{Z}$ define

$$L_n F : \mathcal{A} \to \mathcal{B} \qquad L_n F = H^{-n} \circ LF \circ D^-_{\mathcal{A}} = H^0 \circ T^{-n}_{D(\mathcal{B})} \circ LF \circ D^-_{\mathcal{A}}$$

Notice the negative sign in the translation, this is a slight difference with the right derived functor situation.

Remark 5.73. There is an analogous statement to proposition 5.69 for the left derived functors $L_n F$.

1. $L_n F = 0$ for $n > 0$.
2. $L_0 F$ is right exact.
3. The natural transformation $F \to L_0 F$ is an isomorphism if and only if F is right exact.
4. A short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$ induces a long exact sequence
$$\cdots \to R^1 F(N) \to R^0 F(L) \to R^0 F(M) \to R^0 F(N) \to 0$$
5. Given a projective resolution $\cdots \to P^1 \to P^0 \to M \to 0$, we can compute $L_n F$ by taking cohomology of $C(F)(P)$, where P is the truncated complex $\cdots \to P^1 \to P^0 \to 0 \to \cdots$.
$$L_n F(M) \cong H^n\Big(C(F)(P)\Big)$$

5.5 Derived hom and tensor functors

Let $\mathcal{A}$ be an abelian category with enough injectives, and let $\mathcal{I}$ be the subcategory of injective objects.

Example 5.74 (Ext functors)**.** For $A \in \mathrm{ob}(\mathcal{A})$, consider the covariant left exact functor

$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathrm{AbGp}$$

By the theory developed above, this has a right derived functor

$$R\,\mathrm{Hom}_{\mathcal{A}}(A, -) : D^+(\mathcal{A}) \to D^+(\mathrm{AbGp})$$

Taking cohomology, we obtain the classical right derived functors, known as Ext.

$$\mathrm{Ext}^i_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathrm{AbGp} \qquad \mathrm{Ext}^i_{\mathcal{A}}(A, -) = H^i \circ R\,\mathrm{Hom}_{\mathcal{A}}(A, -)$$

Example 5.75 (Hyperext)**.** We generalize the previous example to replace the object A with a complex $A^\bullet$. Given complexes $A^\bullet, B^\bullet \in \mathrm{ob}(C(\mathcal{A}))$, define

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) := \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A^k, B^{k+n})$$

To make $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ into a complex, define a differential on it by

$$d^n(f) = d_B \circ f - (-1)^n f \circ d_A$$

where d_A, d_B are the respective differentials of $A^\bullet, B^\bullet$. By definition, the n-cocycles of $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ are in bijection with chain maps $A^\bullet \to T^n(B^\bullet)$, where T is the translation functor. Similarly, the n-coboundaries of $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ are nullhomotopic chain maps. So the cohomology is just the group of morphisms in the homotopy category.

$$H^n\left(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)\right) = \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, T^n(B^\bullet))$$

We leave it to the interested reader to verify that $\mathrm{Hom}^\bullet$ defines an exact bifunctor

$$\mathrm{Hom}^\bullet : K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathcal{A})$$

Before discussing the derived functors, we need a lemma. From now on, we drop the upper dots in referring to complexes $A^\bullet, B^\bullet$.

Lemma 5.76. *Let $A \in \mathrm{ob}(K(\mathcal{A})), B \in \mathrm{ob}(K^+(\mathcal{I}))$. Assume that at least one of A, B is acyclic. Then* $\mathrm{Hom}^\bullet(A, B)$ *is acyclic.*

Proof. By the previous discussion, it suffices to show that any morphism $A \xrightarrow{f} B$ is nullhomotopic. If A is acyclic, we proved this earlier in lemma 5.28. If B is acyclic, then by the same lemma the identity $B \to B$ is nullhomotopic, so $f = \mathrm{Id}_B \circ f$ is nullhomotopic. □

We continue our example. By the lemma, the functor

$$\mathrm{Hom}^\bullet(A, -) : K^+(\mathcal{A}) \to K(\mathrm{AbGp})$$

is right adapted to the subcategory $\mathcal{R} = K^+(\mathcal{I})$, so it has a right derived functor. Because this is a bifunctor, it is functorial in the first variable, so we obtain an exact bifunctor

$$R_2\,\mathrm{Hom}^\bullet : K(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathrm{AbGp})$$

The subscript 2 just keeps track of the fact that we derived with respect to the second variable. Now, again using the lemma, $R_2 \operatorname{Hom}^\bullet$ takes quasi-isomorphisms in the first variable to quasi-isomorphisms, so we can also derive with respect to the first variable, and obtain a derived functor

$$R_1 R_2 \operatorname{Hom}^\bullet : D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D^+(\mathrm{AbGp})$$

Now suppose that $\mathcal{A}$ also has enough projectives. By dual arguments, we can obtain a doubly-derived functor

$$R_2 R_1 \operatorname{Hom}^\bullet : D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathrm{AbGp})$$

If $\mathcal{A}$ has both enough injectives and enough projectives, one can show that these functors are canonically isomorphic (when restricted to the intersection of their domains), so we denote it by

$$R \operatorname{Hom} : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathrm{AbGp})$$

This generalizes the classical fact that $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ can be computed using either an injective resolution of B or a projective resolution of A. In most situations, having enough projectives is too much to ask, so in the literature it is common to only assume there are enough injectives and refer to $R_1 R_2 \operatorname{Hom}^\bullet$ by $R \operatorname{Hom}$.

Definition 5.77. Let $R \operatorname{Hom}^\bullet = R_1 R_2 \operatorname{Hom}^\bullet$ as described above. For $A^\bullet \in \operatorname{ob}(D(\mathcal{A}))$ and $B \in \operatorname{ob}(D^+(\mathcal{A}))$, we define the **hyperext groups**

$$\operatorname{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := H^i(R \operatorname{Hom}^\bullet(A^\bullet, B^\bullet))$$

for $i \in \mathbb{Z}$ (though of course they vanish for $i < 0$ as with any right derived functor).

Remark 5.78. We can recover the fact that $\operatorname{Ext}^0_{\mathcal{A}}(A, B) = \operatorname{Hom}(A, B)$ as follows. Let $A^\bullet \in \operatorname{ob}(D(\mathcal{A})), B \in \operatorname{ob}(D^+(\mathcal{A}))$ as above. There exists a quasi-isomorphism

$$s : B^\bullet \to I^\bullet$$

with $I^\bullet \in \operatorname{ob}(K^+(\mathcal{I}))$ a complex of injectives. Then by general properties of right derived functors,

$$R \operatorname{Hom}^\bullet(A^\bullet, B^\bullet) \cong R \operatorname{Hom}^\bullet(A^\bullet, I^\bullet) \cong \operatorname{Hom}^\bullet(A^\bullet, I^\bullet)$$

Then taking cohomology,

$$H^i(R \operatorname{Hom}(A^\bullet, B^\bullet)) \cong H^i\left(\operatorname{Hom}^\bullet(A^\bullet, I^\bullet)\right) \cong \operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, T^i(I^\bullet))$$

Since $I^\bullet$ and $T^i(I^\bullet)$ consist of injectives, there are no quasi-isomorphisms $A^\bullet \to T^i(I^\bullet)$ which are not already isomorphisms in $K(\mathcal{A})$, so

$$\operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, T^i(I^\bullet)) \cong \operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, T^i(I^\bullet)) \cong \operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, T^i(B^\bullet))$$

Putting this all together,

$$\operatorname{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \cong \operatorname{Hom}_{\mathcal{A}}(A^\bullet, T^i(B^\bullet))$$

and for $i = 0$ we have

$$\mathrm{Ext}^0_{\mathcal{A}}(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$$

Since the functor $\mathcal{A} \to D(\mathcal{A})$ is fully faithful, if $A^\bullet = D(A), B^\bullet = D(B)$ are complexes concentrated in degree zero, this gives us

$$\mathrm{Ext}^0_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, B)$$

Example 5.79 (Tor and Hypertor)**.** Let R be a commutative ring and let $\mathcal{A}$ be the category of R-modules. Then $\mathcal{A}$ has enough injectives and enough projectives, and has a tensor product operation. Let $\mathcal{P} \subset \mathcal{A}$ be the full subcategory of projectives. Fix an R-module A. The functor

$$A \otimes - : \mathcal{A} \to \mathcal{A}$$

is left exact (all tensor products will be over R). So there is a left derived functor

$$A \otimes^L - : D^-(\mathcal{A}) \to D^-(\mathcal{A})$$

Taking cohomology, we recover the classical Tor functors.

$$\mathrm{Tor}^R_i(A, -) = H^{-i}(A \otimes^L -)$$

Now as above, we generalize to replace A with a complex. Given complexes $A^\bullet, B^\bullet \in \mathrm{ob}(K(\mathcal{A}))$, define

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{p+q=n} A^p \otimes B^q$$

with differential

$$d^n = d_A \otimes 1 + (-1)^n (1 \otimes d_B)$$

This makes $(A^\bullet \otimes B^\bullet)^\bullet$ into a complex, and we obtain an exact bifunctor

$$- \otimes - : K(\mathcal{A}) \times K(\mathcal{A}) \to K(\mathcal{A})$$

Then we need a parallel to lemma 5.76.

Lemma 5.80. *Let $A^\bullet \in \mathrm{ob}(K^-(\mathcal{A})), B^\bullet \in \mathrm{ob}(K^-(\mathcal{P}))$. If $A^\bullet$ or $B^\bullet$ is acyclic, then so is $A^\bullet \otimes B^\bullet$.*

Proof. Uses spectral sequences, details omitted. □

Now we can continue the example. By the lemma, $K^-(\mathcal{P})$ is left adapted for the functor $A^\bullet \otimes -$, so we get a left derived functor

$$L_2(- \otimes -) : K^-(\mathcal{A}) \times D^-(\mathcal{A}) \to D(\mathcal{A})$$

and with another application of the lemma we can derived with respect to the first variable and obtain

$$L_1 L_2(- \otimes -) : D^-(\mathcal{A}) \times D^-(\mathcal{A}) \to D(\mathcal{A})$$

As before, we can do these derivations in the reverse order, and the resulting functors are canonically isomorphic. So we denote it by

$$- \otimes^L - : D^-(\mathcal{A}) \to D^-(\mathcal{A}) \to D(\mathcal{A})$$

Then hypertor groups are defined analogously to hyperext groups.

As in the previous example, let R be a commutative ring and $\mathcal{A}$ be the category of R-modules. Recall the classical tensor-hom adjunction, which we express as a natural isomorphism

$$\operatorname{Hom}_R(M, \operatorname{Hom}_R(N, P)) \cong \operatorname{Hom}_R(M \otimes_R N, P)$$

By "natural," we mean that this is functorial in the variables M and P. (Perhaps this is functorial in N as well, I forget.) Anyway, in the derived setting there is a generalization of this using derived hom and derived tensor.

Proposition 5.81. *For complexes $A^\bullet, B^\bullet \in \operatorname{ob}(D^-(\mathcal{A})), C^\bullet \in \operatorname{ob}(D^+(\mathcal{A}))$, there is a natural isomorphism*

$$\operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, R\operatorname{Hom}^\bullet(B^\bullet, C^\bullet)) \cong \operatorname{Hom}_{D(\mathcal{A})}\left(A^\bullet \otimes^L B^\bullet, C^\bullet\right)$$

Note that this does not give rise to a new proof of the classical tensor-hom adjunction, since the classical adjunction is used as a setp in the proof.

Remark 5.82. In algebraic geometry, the common abelian category of interest is the category $\mathcal{A}$ of $\mathcal{O}_X$-modules, where X is a scheme and $\mathcal{O}_X$ is the structure sheaf on X. This category has enough injectives, but in general does not have enough projectives.

One remedy or workaround for the lack of projectives is to utilize flat $\mathcal{O}_X$-modules instead. So even though projectives do not form a category adapted for tensor project of $\mathcal{O}_X$-modules, flat modules do, so there is still a derived tensor functor.

5.6 Derived sheaf-theoretic functors

Let X be a topological space, and let $\operatorname{Sh}(X)$ be the abelian category of sheaves of abelian groups on X. Recall that $\operatorname{Sh}(X)$ has enough injectives.

Example 5.83 (Sheaf cohomology and hypercohomology)**.** Let $\Gamma : \operatorname{Sh}(X) \to \operatorname{AbGp}$ be the global sections functor, $\Gamma(F) = F(X)$ is the group of global sections. Then Γ is left exact, so it has a right derived functor

$$R\Gamma : D^+(\operatorname{Sh}(X)) \to D^+(\operatorname{AbGp})$$

So taking cohomology, we recover the classical sheaf cohomology groups.

$$R^i\Gamma(F) = H^i(X, F)$$

for a sheaf F on X. When $F^\bullet \in \operatorname{ob}(D^+(\operatorname{Sh}(X))$ is a complex of sheaves, the groups

$$H^i(X, F^\bullet) = R^i\Gamma(F^\bullet)$$

are called **hypercohomology groups** of $F^\bullet$.

Example 5.84 (Higher direct images)**.** First we recall the direct image functor. Let $f : X \to Y$ be a continuous map of topological spaces. The direct image functor

$$f_* : \operatorname{Sh}(X) \to \operatorname{Sh}(Y)$$

takes a sheaf F on X to a new sheaf f_*F on Y defined on an open subset $U \subset Y$ by

$$f_*F(U) = F(f^{-1}(U))$$

The functor f_* is left exact, so it has a right derived functor

$$Rf_* : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y))$$

Taking cohomology, we obtain the **higher direct image** functors.

$$R^if_* := H^i \circ Rf_*$$

Definition 5.85. Let $F \in \mathrm{Sh}(X)$ be a sheaf, and $U \subset X$ an open subset. Let $s \in F(U)$ be a section. The **support** of s is

$$\mathrm{supp}(s) := \{u \in U : s_u \neq 0\}$$

where s_u is the image of s in the stalk F_u.

Definition 5.86. A continuous map $g : X \to Y$ is **proper** if for any compact set in Y, the preimage in X is compact.

Example 5.87 (Higher direct image with compact support)**.** Let X, Y be locally compact spaces, and let $f : X \to Y$ be a continuous map. For $U \subset Y$ and $F \in \mathrm{Sh}(X)$, define

$$f_!F(U) = \left\{ s \in F(f^{-1}(U)) : \mathrm{supp}(s) \xrightarrow{f} U \text{ is a proper map} \right\}$$

If you are familiar with extension by zero, you might think of $f_!$ as a "relative" version of that. In any case, it is clear that $f_!(F)(U) \subset f_*F(U)$, so you can also think of it as the direct image with "compact support," which is encoded by the condition that $f : \mathrm{supp}(s) \to U$ is proper. It is possible to show that

1. $f_!F$ is a sheaf, in particular a subsheaf of f_*F
2. $F \mapsto f_!F$ is a left exact functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$

The subscript exclamation point is read as "shriek," so the functor $f_!$ is called "f lower shriek." Is is also called the **direct image with compact support**.

An important special case is when $Y = \{*\}$ is a single point. Then $\mathrm{Sh}(Y)$ is just the category of abelian groups [20], and for any sheaf $F \in \mathrm{Sh}(X)$, $f_!F(X)$ is the group of global sections $s \in F(X)$ such that $\mathrm{supp}(s)$ is a compact subset of X. (This is probably where the naming comes from.) This group is also denoted

$$\Gamma_c(X, F) = f_!F(X)$$

The subscript c here is short for "compact." Since $\mathrm{Sh}(X)$ has enough injectives, $f_!$ has a right derived functor. However, before discussing this derived functor, we note that is common

[20] A sheaf on a single point space is just described by the group of global sections.

when working with $f_!$ to enlarge the adapted subcategory from injective sheaves to **soft** sheaves [21]. Injective sheaves are soft, and soft sheaves form a right adapted subcategory for $f_!$, so now we move on to discussing its right derived functor.

$$Rf_! : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y))$$

Taking cohomology, we obtain functors

$$R^i f_! = H^i \circ Rf_!$$

which are called **higher direct images with compact support**. When $Y = *$ is a point, the derived functor is denoted

$$R\Gamma_c : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{AbGp})$$

and the composition with cohomology is denoted by

$$H^i_c(X, F) = H^i \circ R\Gamma_c(F)$$

Definition 5.88. Let X be a locally compact space. The **cohomological dimension** of X is the smallest integer n such that for any sheaf $F \in \mathrm{Sh}(X)$, $H^i_c(X, F) = 0$ for all $i > n$. We denote it by $\dim_C X$.

Example 5.89. I'm pretty sure that if X is a smooth manifold, then the cohomological dimension agrees with the dimension of the local charts.

As a final example, we can state (but definitely not prove) a version of Verdier duality.

Theorem 5.90 (Verdier duality)**.** *Let $f : X \to Y$ be a continuous map between locally compact spaces of finite cohomological dimension. There exists a functor*

$$f^! : D^+(\mathrm{Sh}(Y)) \to D^+(\mathrm{Sh}(X))$$

[22] *and a natural isomorphism*

$$R\,\mathrm{Hom}^\bullet(Rf_! F^\bullet, G^\bullet) \cong R\,\mathrm{Hom}(F^\bullet, f^! G^\bullet)$$

By "natural," we mean functorial in $F^\bullet$ and $G^\bullet$.

Remark 5.91. If we apply the functor H^0 to both sides of the isomorphisma bove, we obtain

$$\mathrm{Hom}_{D(\mathrm{Sh}(Y))}(Rf_! F^\bullet, G^\bullet) \cong \mathrm{Hom}_{D(\mathrm{Sh}(X))}(F^\bullet, f^! G^\bullet)$$

which is again functorial in $F^\bullet, G^\bullet$. So $f^!$ is right adjoint to $Rf_!$.

[21]A sheaf F is **soft** if for any closed set $K \subset X$, any section $s \in F(K)$ extends to a section $\widetilde{s} \in F(X)$. Informally, this says that the restriction map $F(X) \to F(K)$ is surjective, but technically speaking that doesn't make any sense, since restriction is only defined for open subsets. But if working with a sheaf of functions, this makes sense, and there are ways to make this precise.

[22]This functor is read as "f upper shriek."

Remark 5.92. In general, $f^!$ only exists on the level of derived categories. That is to say, there is not a functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ which has $f^!$ as its derived functor (maybe there is in some cases, if you add more assumptions on X, Y, but not in general).

Definition 5.93. Take $Y = \{*\}$ in the theorem, so $\mathrm{Sh}(Y) = \mathrm{AbGp}$. Let $\mathbb{Z}$ denote the complex with $\mathbb{Z}$ concentrated in degree zero in $D^+(\mathrm{Sh}(Y)) = D^+(\mathrm{AbGp})$, and let $f : X \to \{*\}$ be the unique map. Define

$$\mathcal{D}_X^\bullet = f^!(\mathbb{Z}) \in D^+(\mathrm{Sh}(X))$$

This is called the **dualizing complex** of X. Verdier duality says that

$$R\,\mathrm{Hom}^\bullet(R\Gamma_c(X, F^\bullet), \mathbb{Z}) \cong R\,\mathrm{Hom}^\bullet(F^\bullet, \mathcal{D}_X^\bullet)$$

In general, there isn't too much more one can say about $\mathcal{D}_X^\bullet$. In particular, even though $\mathbb{Z}$ is concentrated in a single degree, $\mathcal{D}_X^\bullet$ need not be.

Remark 5.94. If X is a smooth manifold, one can prove that $\mathcal{D}_X^\bullet$ is concentrated in degree zero. Using this fact, one can recover classical Poincaré duality from Verdier duality, using some additional analysis of $\mathcal{D}_X^\bullet$. So Verdier duality provides a vast generalization of Poincaré duality, since it applies not only to smooth manifolds, but to the much larger category of locally compact topological spaces.